Homework 2

Saturday, April 3, 2021 10:13 AM

The book will refer to Nisheeth Vishnoi's book.

Problem 1: Exercise 10.2 in the book. Parts (a), (b) & (c). The total bit complexity is the total number of bits required to represent all the numerators & denominators of the rational numbers in the input.

Problem 2: Exercise 10.4 in the book. Parts (a) & (b).

Problem 3: Let $F : \text{int}(K) \to \mathbb{R}$ be a self-concordant barrier function with complexity parameter ν. Prove that

$K \subseteq \mathbb{R}^n$ is a convex set.

(a) F satisfies the Newton-Local condition with $\delta_0 = 1/6$.

(b) $\forall x \in \text{int}(K)$ & $y \in K$, it holds that $\langle \nabla F(x), y - x \rangle \leq \nu$.

(c) Given an initial $\eta_0 > 0$ & $x_0 \in \text{int}(K)$ s.t. $\| N_{\eta_0}(x_0) \|_{x_0} \leq 1/6$,

devise an interior point method algorithm to solve the following optimization problem:

$$\min_{x \in K} \langle c, x \rangle$$

The algorithm should run in $O\left(\sqrt{\nu} \log\left(\frac{\nu}{\varepsilon \eta_0}\right)\right)$ iterations & output a point $\hat{x} \in \text{int}(K)$

$$\langle c, \hat{x} \rangle - \langle c, x^* \rangle \leq \varepsilon.$$

(each iteration should be essentially solving a linear system)

Below are detailed steps to solve Problem 3 (you can try solving on your own if you wish)

(a) We will prove a slightly stronger statement.
$\forall\ x, y$ s.t. $\|y - x\|_x < 1$, it holds that

$$(1 - \|y-x\|_x)^2 H_F(x) \preceq H_F(y) \preceq (1 - \|y-x\|_x)^{-2} H_F(x)$$

$\forall$ x, y s.t. $\|y-x\|_x < 1$, it holds that

$$\left(1 - \|y-x\|_x\right)^2 H_F(x) \preceq H_F(y) \preceq \left(1 - \|y-x\|_x\right)^{-2} H_F(x)$$

Towards this, define $\alpha(t) = \|y-x\|^2_{x+t(y-x)}$

Prove that it satisfies the following differential condition

$$\frac{d}{dt}\left(\frac{1}{\sqrt{\alpha(t)}}\right) \geq -1$$

Using this to deduce the properties of $\beta(t) = \|u\|^2_{x+t(y-x)}$ for an arbitrary vector u & finally deduce the NL condition via the above stronger condition.

(b) Define $\alpha(t) = \langle \nabla F(x+t(y-x)), y-x \rangle$. Prove that $\alpha(t)$ satisfies the following differential condition

$$\frac{d}{dt}\left(-\frac{1}{\alpha(t)}\right) \geq \frac{1}{\nu}$$

From this deduce the statement.

(c) The main difference from the proof for linear programming would be how to bound

$$\langle c, \hat{x} - x_{\eta^*_T} \rangle$$

Please devise a different way of bounding this as the method discussed in the lecture does not work in general. Ofcourse also fill in the rest of the details.