Sampling from a Convex Body

Based on

1. Ravi Kannan's notes.
2. Jonathan Kelner's lecture notes.
3. Lap Chi Lau's lecture notes.
4. "Techniques in Optimization and Sampling" by Yin Tat Lee and Santosh Vempala.
5. "Algorithmic Convex Geometry" by Santosh Vempala.
6. "Geometric Random Walks" by Santosh Vempala.

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Recall that for a random variable X, $Y := \Sigma^{-1/2}(X - \mathbb{E}[X])$ is isotropic.

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- For $i = 1$ to $n \log R$, do
  1. Use the ball walk from x to sample N random points $x_1, \ldots, x_N$ from $A_i K_i$.
  2. Compute $C := \frac{1}{N}\sum_{i\in[N]}\left(x_i - \frac{1}{N}\sum_i x_i\right)\left(x_i - \frac{1}{N}\sum_i x_i\right)^T$ and set $A_{i+1} := C^{-1/2}A_i$.
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Lemma

If K' is isotropic, then with $N = \mathrm{poly}(n)$, the matrix $C := (\sum_{i \in [N]} x_i x_i^T)/N$ satisfies $\|C - I\| \leq 0.5$ w.h.p.

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Theorem

For a convex body K in isotropic position, we have

$$\sqrt{\frac{n+2}{n}}\, \mathcal{B}(0,1) \subseteq K \subseteq \sqrt{n(n+2)}\, \mathcal{B}(0,1)\,.$$

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Therefore, $R/r = O(n)$.

Bounding conductance

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Let K be a convex body of diameter D such that $\mathcal{B}(0,1) \subseteq K$ and $\ell(u) \geq \ell \ \forall u \in K$. Then conductance of the ball walk with step size δ is

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Let $K = S_1 \cup S_2$ be a partition into measurable sets. We will prove that

$$\int_{S_1} P_x(S_2)dx = \Omega\left(\frac{\ell^2\delta}{\sqrt{n}D}\right) \min\left\{\mathrm{vol}\left(S_1\right), \mathrm{vol}\left(S_2\right)\right\}.$$

Let $S_1' \subseteq S_1$ be the set of points from which the random walk is unlikely to leave S_1.

$$S_1' := \left\{ x \in S_1 : P_x(S_2) < \frac{\ell}{4} \right\} \qquad \text{and} \qquad S_2' := \left\{ x \in S_2 : P_x(S_1) < \frac{\ell}{4} \right\}.$$

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$$\int_{S_1} P_x(S_2)dx \geq \frac{\ell}{4}\mathrm{vol}\left(S_1 \setminus S_1'\right) \geq \frac{\ell}{8}\mathrm{vol}\,(S_1)\,.$$

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- Therefore, assume that $\mathrm{vol}\,(S_1') \geq \mathrm{vol}\,(S_1)/2$ and $\mathrm{vol}\,(S_2') \geq \mathrm{vol}\,(S_2)/2$.
- For any $u \in S_1'$ and $v \in S_2'$

$$d_{\mathrm{TV}}\left(P_u, P_v\right) \geq 1 - P_u(S_2) - P_v(S_1) > 1 - \frac{\ell}{2}.$$

Lemma (One-step overlap)

Let $u', v' \in K$ *such that* $\ell(u'), \ell(v') \geq \ell$ *and* $\|u' - v'\| \leq t\delta/\sqrt{n}$*. Then* $d_{\mathrm{TV}}\left(P_{u'}, P_{v'}\right) \leq 1 + t - \ell$*.*

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$$\mathrm{vol}\left(S_3\right) \geq \frac{2d(S_1', S_2')}{D} \min\left\{\mathrm{vol}\left(S_1'\right), \mathrm{vol}\left(S_2'\right)\right\} \geq \frac{\ell\delta}{2\sqrt{n}D} \min\left\{\mathrm{vol}\left(S_1\right), \mathrm{vol}\left(S_2\right)\right\}.$$

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$$\begin{aligned}
\int_{S_1} P_x(S_2)dx &= \frac{1}{2}\int_{S_1} P_x(S_2)dx + \int_{S_2} P_x(S_1)dx \\
&\geq \frac{1}{2}\left(\frac{\ell}{4}\mathrm{vol}\left(S_3\right)\right) \geq \frac{\ell^2\delta}{16\sqrt{n}D} \min\left\{\mathrm{vol}\left(S_1\right), \mathrm{vol}\left(S_2\right)\right\}.
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If $\mathcal{B}(0,1) \subseteq K$ then, $\mathrm{vol}(K + \alpha\mathcal{B}(0,1)) \leq (1+\alpha)^n \,\mathrm{vol}(K)$ (verify).

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Claim: For $\alpha \geq \delta\sqrt{n}$, all points in $\mathcal{B}(\alpha e_1, \delta)$ with $x_1 \leq \alpha - \delta/\sqrt{n}$ belong to $\mathcal{B}(0, \alpha)$.

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- Therefore,

$$\sum_{i=1}^n x_i^2 \le \left(\alpha - \frac{\delta}{\sqrt{n}}\right)^2 + \sum_{i=2}^n x_i^2 \le \alpha^2 - \frac{2\alpha\delta}{\sqrt{n}} + \frac{\delta^2}{n} + \delta^2 - \frac{\delta^2}{n} \le \alpha^2.$$

Therefore,

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Using standard estimates, $\ell(u) = \Omega(1)$.

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Membership oracle for K'.

- Given x, is there a $y \in K$ such that $\|y - x\| \leq \alpha$.
- Can be solved using the ellipsoid algorithm and membership oracle for K.

Generating a warm start

Mixing time is $O\left(\frac{\log(M/\epsilon)}{\phi^2}\right)$, where $M := \sup_A Q_0(A)/Q(A)$.

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Since $\text{vol}\,(K_{i+1}) \leq e\ \text{vol}\,(K_i)$, a random point from K_i is a e-warm start for K_{i+1}.

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Note that

$$\mathbb{E}_{x\sim f_i}\left[\frac{f_{i+1}(x)}{f_i(x)}\right] = \int \frac{f_{i+1}(x)}{f_i(x)} \cdot \frac{f_i(x)}{\int f_i} dx = \frac{\int f_{i+1}}{\int f_i}.$$

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Choose $\{f_i\}$ such that samples from f_i provide a warm start for sampling from f_{i+1}, and f_0 is easy to sample from.

Surface area

The surface area of a body K is defined as

$$\mathrm{vol}\left(\partial K\right) := \lim_{\epsilon \to 0} \frac{\mathrm{vol}\left(K + \epsilon \mathcal{B}\left(0,1\right)\right) - \mathrm{vol}\left(K\right)}{\epsilon}$$

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Theorem

For a convex body K of diameter D and $S \subseteq K$, we have

$$\mathrm{vol}\left(K \cap \partial S\right) \geq \frac{2}{D} \min\left\{\mathrm{vol}\left(S\right), \mathrm{vol}\left(K \setminus S\right)\right\}.$$

Brunn-Minkowski inequality

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Theorem (Brunn-Minkowski inequality)

Let $A, B \subset \mathbb{R}^n$ be compact measurable sets. Then $\operatorname{vol}(A+B)^{1/n} \geq \operatorname{vol}(A)^{1/n} + \operatorname{vol}(B)^{1/n}$.

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$$\begin{aligned} \mathrm{vol}\,(\lambda A + (1-\lambda)B) &\geq \left(\mathrm{vol}\,(\lambda A)^{1/n} + \mathrm{vol}\,((1-\lambda)B)^{1/n}\right)^n \\ &= \left(\lambda \mathrm{vol}\,(A)^{1/n} + (1-\lambda)\mathrm{vol}\,(B)^{1/n}\right)^n \\ &\geq \left(\mathrm{vol}\,(A)^{\lambda/n}\,\mathrm{vol}\,(B)^{(1-\lambda)/n}\right)^n \\ &= \mathrm{vol}\,(A)^{\lambda}\,\mathrm{vol}\,(B)^{(1-\lambda)}\,. \end{aligned}$$

Proof for cuboids

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$$\operatorname{vol}(A) = \Pi_{i\in[n]} a_i, \quad \operatorname{vol}(B) = \Pi_{i\in[n]} b_i, \quad \operatorname{vol}(A+B) = \Pi_{i\in[n]}(a_i + b_i).$$

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$$\mathrm{vol}\,(A) = \Pi_{i \in [n]} a_i, \quad \mathrm{vol}\,(B) = \Pi_{i \in [n]} b_i, \quad \mathrm{vol}\,(A + B) = \Pi_{i \in [n]} (a_i + b_i).$$

$$\begin{aligned}
\frac{\mathrm{vol}\,(A)^{1/n} + \mathrm{vol}\,(B)^{1/n}}{\mathrm{vol}\,(A+B)^{1/n}} &= \frac{\left(\Pi_{i \in [n]}(a_i)\right)^{1/n} + \left(\Pi_{i \in [n]}(b_i)\right)^{1/n}}{\left(\Pi_{i \in [n]}(a_i + b_i)\right)^{1/n}} \\
&= \left(\Pi_{i \in [n]} \frac{a_i}{a_i + b_i}\right)^{1/n} + \left(\Pi_{i \in [n]} \frac{b_i}{a_i + b_i}\right)^{1/n} \\
&\leq \sum_{i \in [n]} \frac{a_i}{a_i + b_i} + \sum_{i \in [n]} \frac{b_i}{a_i + b_i} \qquad (\mathrm{GM} \leq \mathrm{AM}) \\
&= 1.
\end{aligned}$$

Finite unions of cuboids

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- Note that $(A^+ + B^+) \cap (A^- + B^-) = \emptyset$ and

$$\frac{\mathrm{vol}\,(B)}{\mathrm{vol}\,(A)} = \frac{\mathrm{vol}\,(B^+)}{\mathrm{vol}\,(A^+)} = \frac{\mathrm{vol}\,(B^-)}{\mathrm{vol}\,(A^-)}.$$

$$
\begin{aligned}
\operatorname{vol}(A+B) &\geq \operatorname{vol}\left(A^{+}+B^{+}\right)+\operatorname{vol}\left(A^{-}+B^{-}\right) \\
&\geq \left(\operatorname{vol}\left(A^{+}\right)^{1/n}+\operatorname{vol}\left(B^{+}\right)^{1/n}\right)^{n}+\left(\operatorname{vol}\left(A^{-}\right)^{1/n}+\operatorname{vol}\left(B^{-}\right)^{1/n}\right)^{n} \\
&= \operatorname{vol}\left(A^{+}\right)\left(1+\frac{\operatorname{vol}(B)^{1/n}}{\operatorname{vol}(A)^{1/n}}\right)^{n}+\operatorname{vol}\left(A^{-}\right)\left(1+\frac{\operatorname{vol}(B)^{1/n}}{\operatorname{vol}(A)^{1/n}}\right)^{n} \\
&= \operatorname{vol}(A)\left(1+\frac{\operatorname{vol}(B)^{1/n}}{\operatorname{vol}(A)^{1/n}}\right)^{n}=\left(\operatorname{vol}(A)^{1/n}+\operatorname{vol}(B)^{1/n}\right)^{n}
\end{aligned}
$$

$$
\begin{aligned}
\operatorname{vol}(A+B) &\geq \operatorname{vol}\left(A^{+}+B^{+}\right)+\operatorname{vol}\left(A^{-}+B^{-}\right) \\
&\geq \left(\operatorname{vol}\left(A^{+}\right)^{1/n}+\operatorname{vol}\left(B^{+}\right)^{1/n}\right)^{n}+\left(\operatorname{vol}\left(A^{-}\right)^{1/n}+\operatorname{vol}\left(B^{-}\right)^{1/n}\right)^{n} \\
&= \operatorname{vol}\left(A^{+}\right)\left(1+\frac{\operatorname{vol}(B)^{1/n}}{\operatorname{vol}(A)^{1/n}}\right)^{n}+\operatorname{vol}\left(A^{-}\right)\left(1+\frac{\operatorname{vol}(B)^{1/n}}{\operatorname{vol}(A)^{1/n}}\right)^{n} \\
&= \operatorname{vol}(A)\left(1+\frac{\operatorname{vol}(B)^{1/n}}{\operatorname{vol}(A)^{1/n}}\right)^{n} = \left(\operatorname{vol}(A)^{1/n}+\operatorname{vol}(B)^{1/n}\right)^{n}
\end{aligned}
$$

Any measurable set can be approximated arbitrarily well using unions of cuboids.
General case can be proved using this.