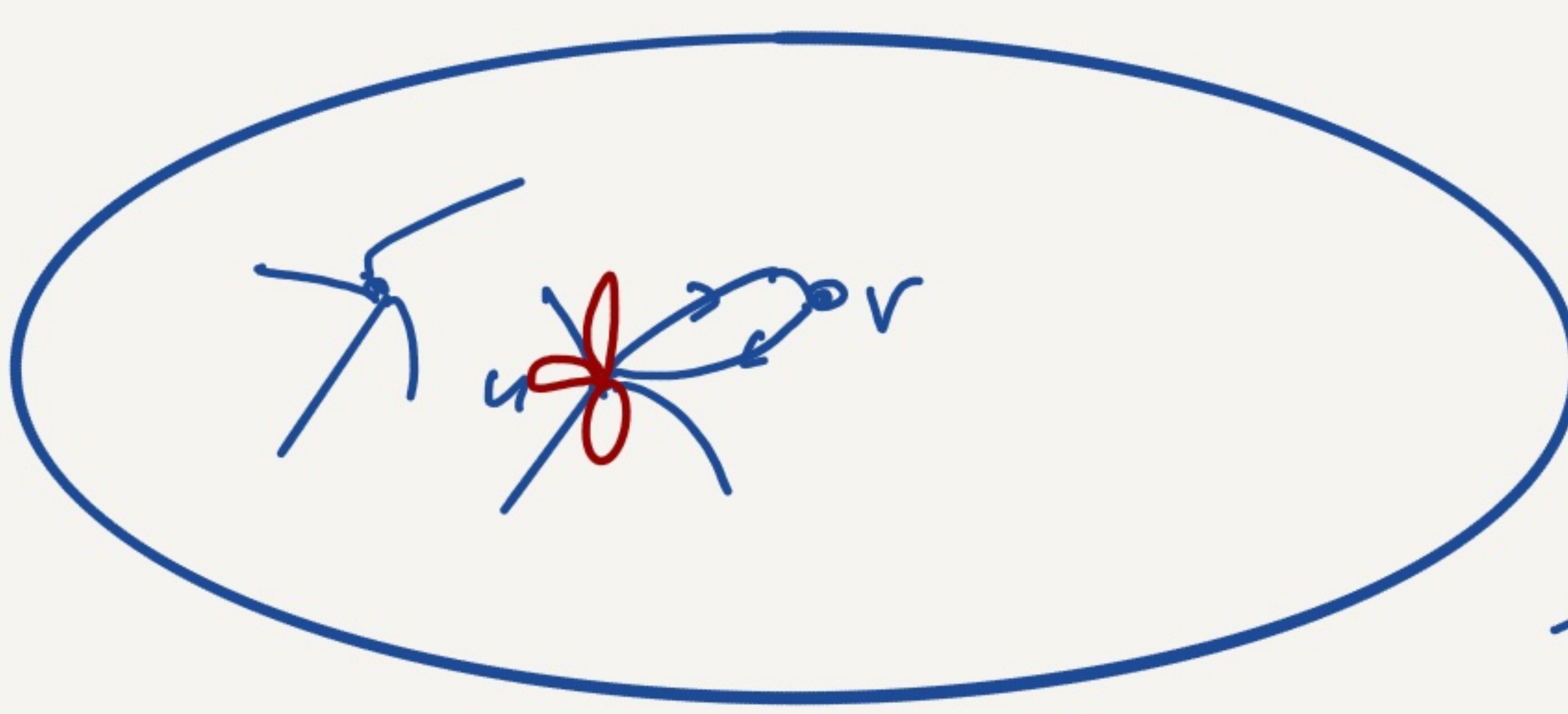
Lovasz - Simonovits tool :

Punchline : Highly versatile tool useful in analyzing convergence rates of r(walks

Setup :

$G = (V, E)$ $\qquad$ $|E| = 2m$



μ : stationary distrib

$\mu(z) \propto dg(z)$

For $z \in V$

For $z' \in N(z)$

Prob mass sent from z to z'

at μ : $\dfrac{\mu(z)}{dg(z)}$

μ induces a distrib on edges $\doteq$ Unif on edges

Take $\vec{p}$ supported on V

$\vec{p}$ induces a distrib on E

depends only on u $\rightarrow$ $\vec{\rho}(u,v) = \dfrac{p(u)}{d_G(u)} = \rho(u,v')$

$\forall\ v, v' \in N(u)$

KEY: Define a potential function which you track to prove guarantees abt convergence rates

$\underline{I}$, h, $\underline{\underline{g}}$

LS potential function $\equiv$ Greedy aggregation rule

let $\vec{p}$ be a distrib on V

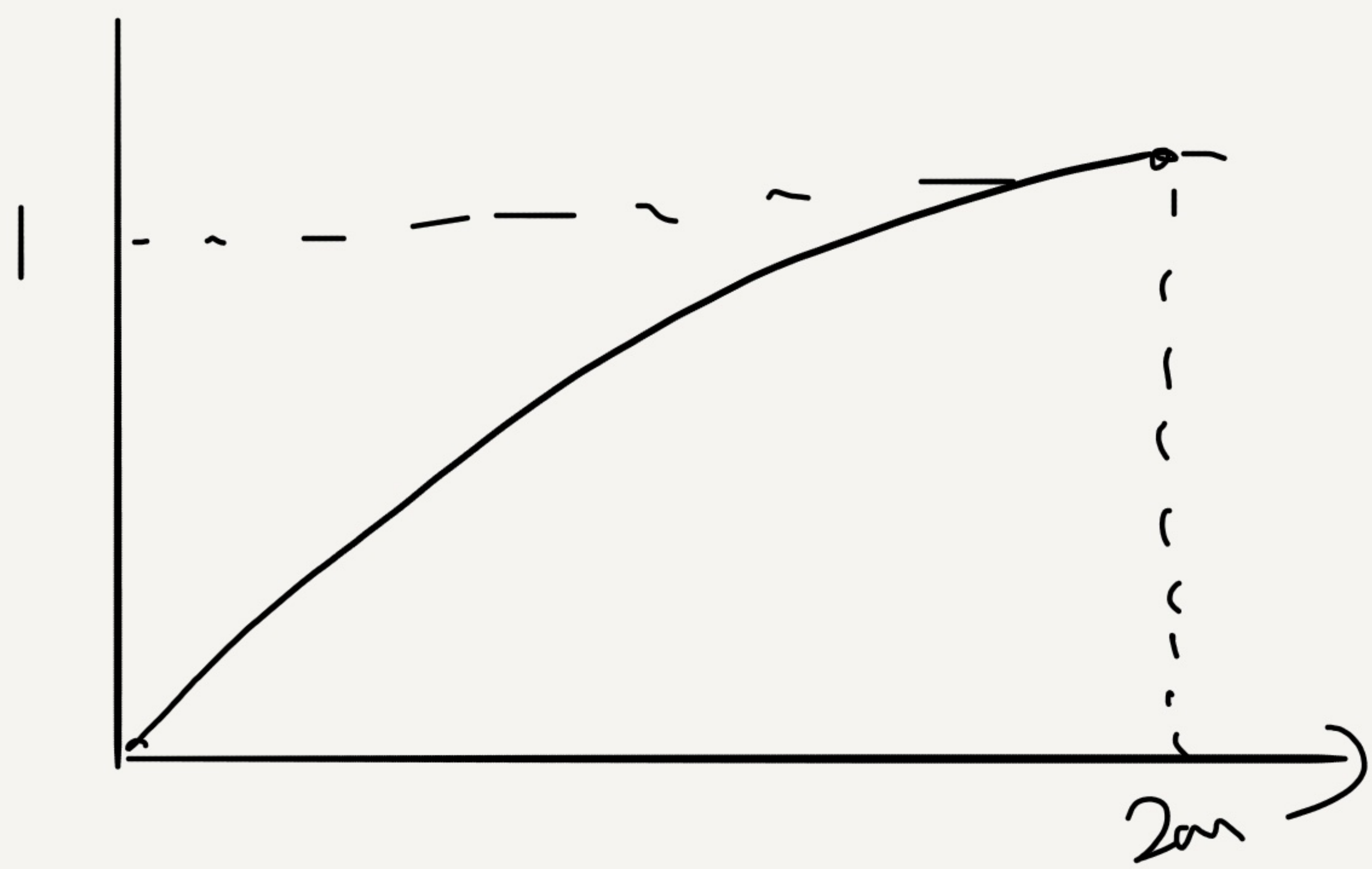
& ρ denote the induced distrib on E

$$\boxed{\rho(e_1) \geq \rho(e_2) \cdots \geq \rho(e_{2m})}$$

$g_p : [0, 2m] \longrightarrow [0, 1]$

I define g_p by specifying values at integer pts $\{0, 1, 2 - - - 2m\}$

$$g_p(x) = \sum_{i=1}^{x} \rho(e_i)$$



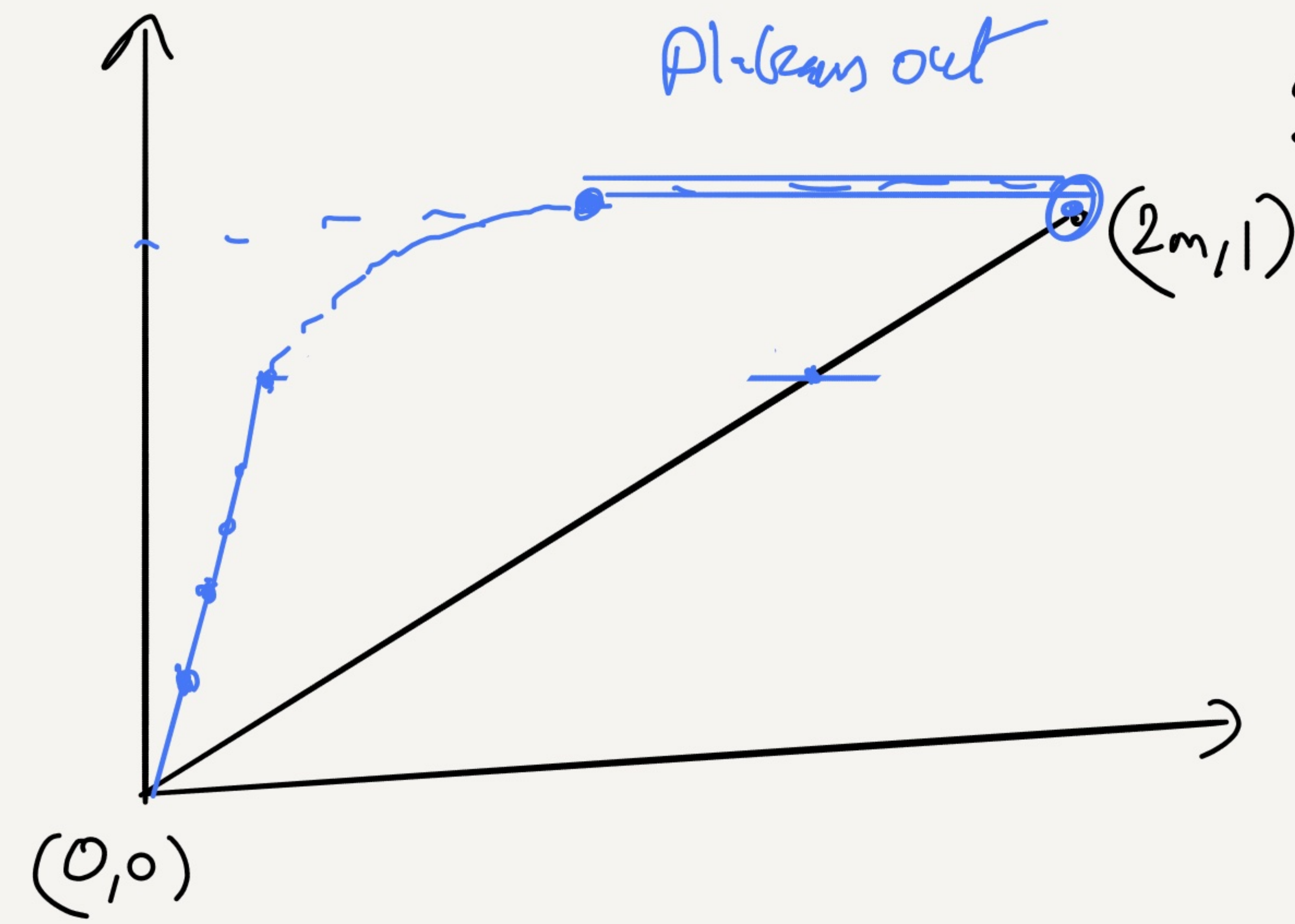
$g_p(0) = 0$

$g_p(2m) = 1$

g_p increases

Sps $p = u$ = stationary distrib

$g_u(x) = \frac{x}{2m}$



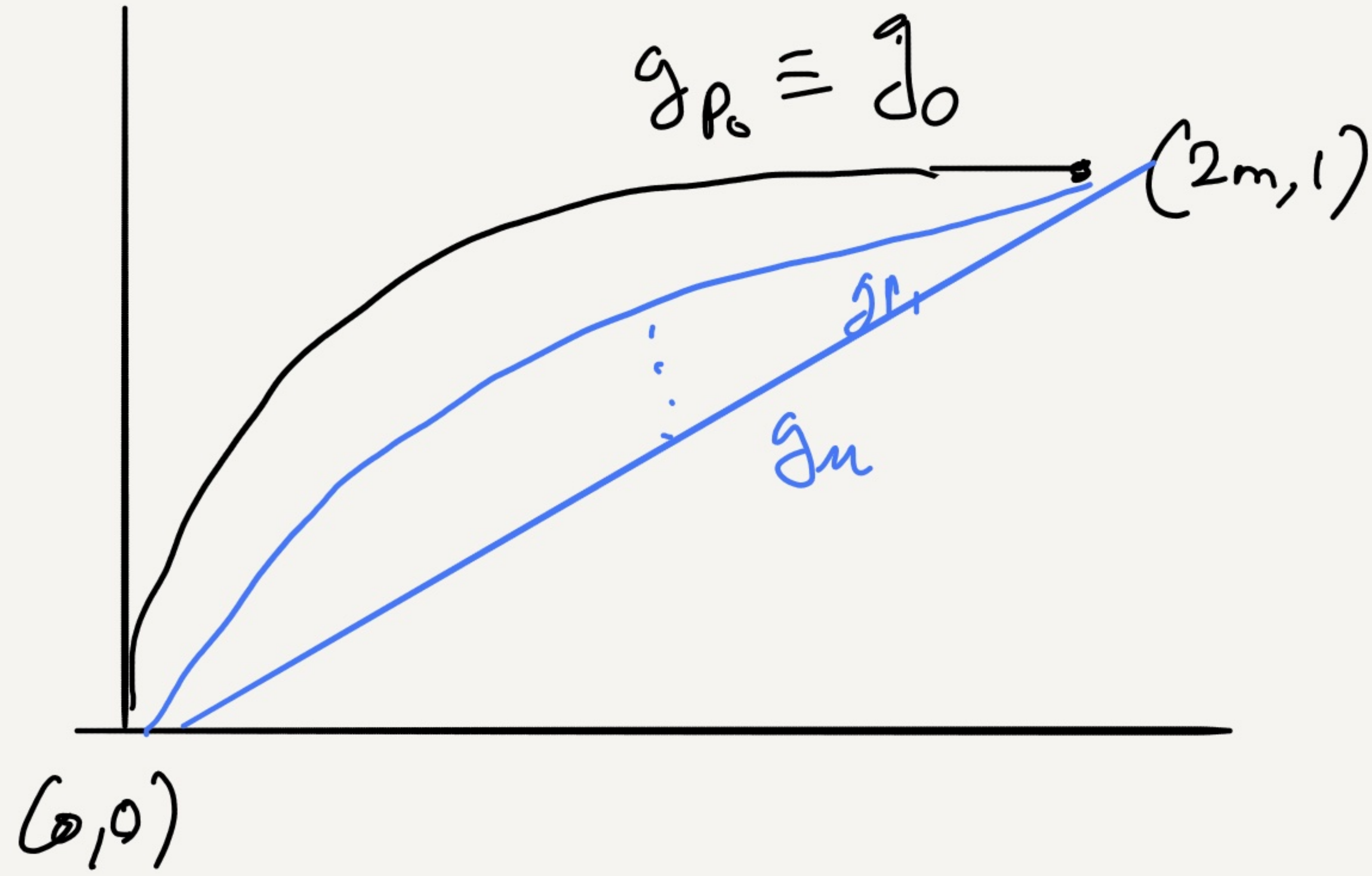
RMK: g_p is concave

Q: Is it conceivable

$$g_{p_t} \xrightarrow{t \to \infty} g_u$$ (YES) by Lov-Simon

$$p_t = M^t \cdot p_0$$

lazy r/walk matrix



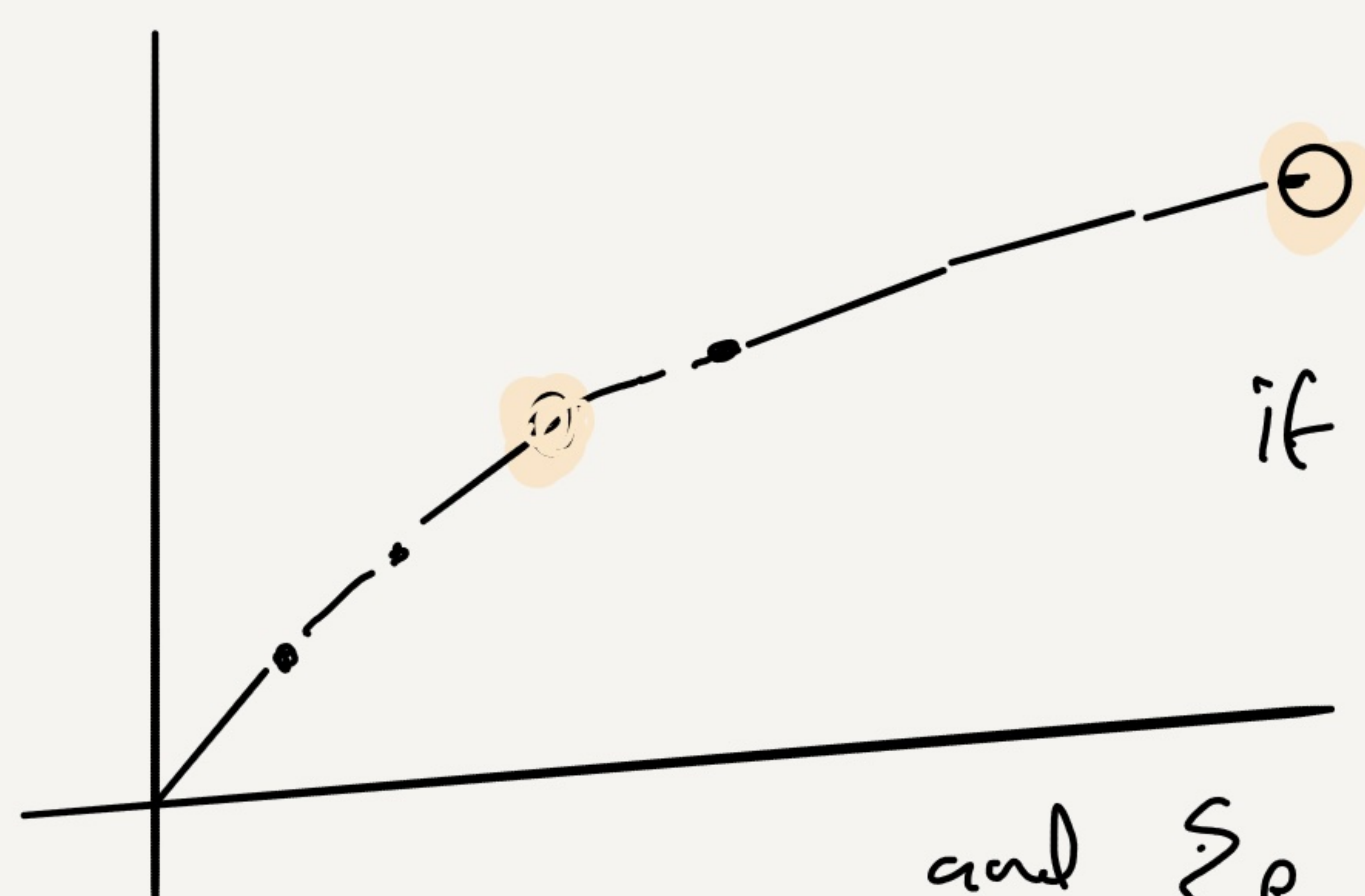
Claim: For every $t \geq 1$

For every $x \in [0, 2m]$

$$g_t(x) \leq g_{t-1}(x)$$

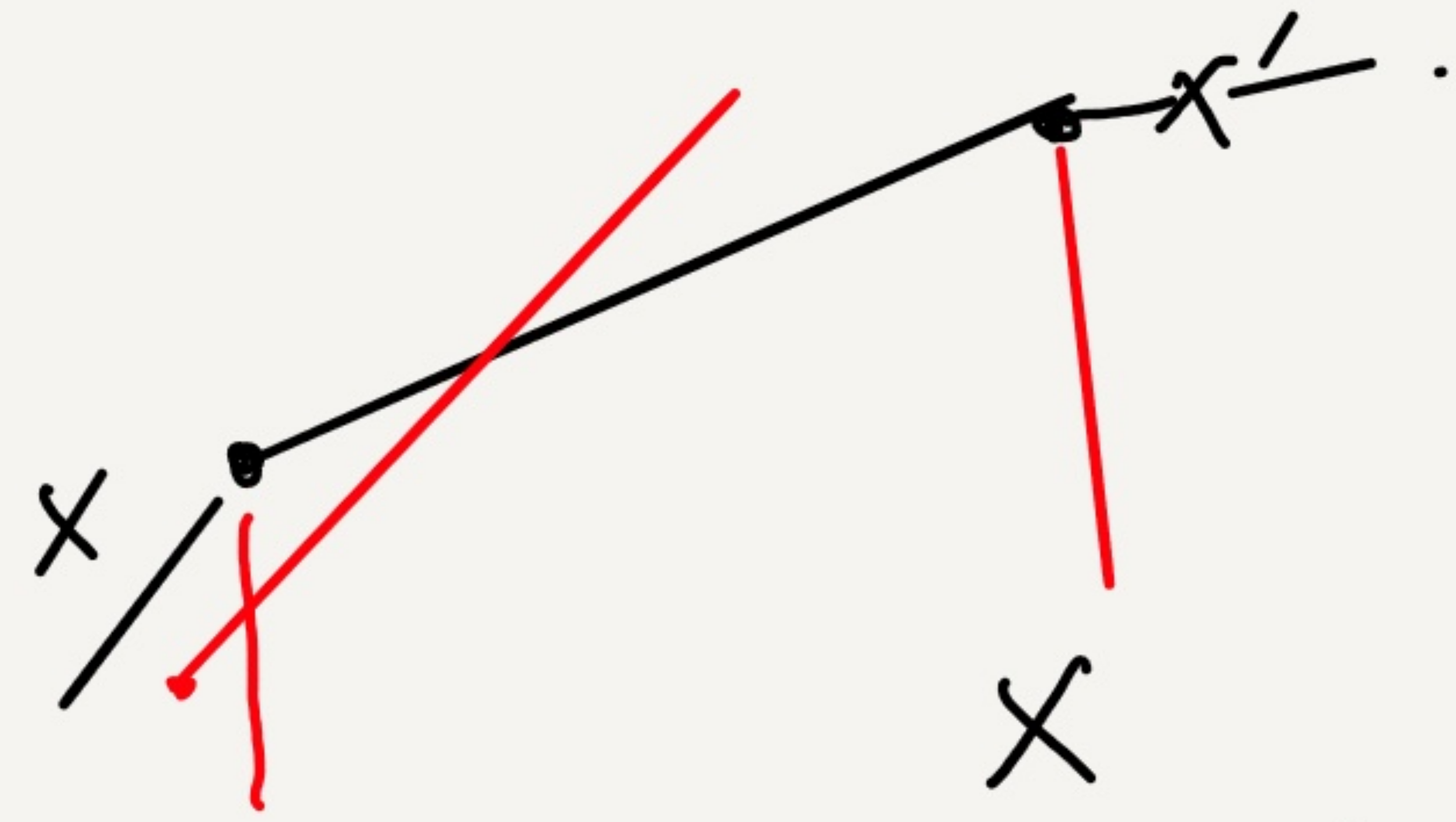
Pf: Hinge points of the t-step curve

$x \in \{0 \dots 2m\}$ is a hinge pt if $\exists S \subseteq V$ st $\text{vol}(S) = x$ and $\{e_1, e_2 \dots e_x\}$



To show $g_t(x) \le g_{t+1}(x)$

it suffices to do this for hinge points x on $\overrightarrow{p^t}$.



$$
\begin{aligned}
g_t(x) &= \sum_{i=1}^{x} \rho^t(e_i) \\
&= \sum_{i=1}^{x} \rho^t(u_i, v_i) \\
&= \sum_{u \in S} p^t(u) \\
&= \sum_{i=1}^{x} \rho^{t+1}(v_i, u_i) \\
&\le g_{t+1}(x) \quad \text{(By greedy rule)}
\end{aligned}
$$

An alternate characterization of $g_t(x)$:

$$g_t(x) = \boxed{\begin{array}{c} \max \ \vec{c} \cdot \vec{p}^{\,t} \\ \text{st} \\ 0 \le c_e \le 1 \\ \sum c_e = X \end{array}} =$$

LS[Thm] : For every distrib $\vec{p^0}$

For every t

we have

① If $x \le m$ then

$$g_t(x) \le \frac{g_{t-1}(x - 2\phi x) + g_{t-1}(x + 2\phi x)}{2}$$

② If $x > m$ then

$$-g_t(x) \le \frac{g_{t-1}(x - 2\phi(2m - x)) + g_{t-1}(\ )}{2}$$

LS [Thm 2]:

For every distrib $\vec{p_0}$

every $t \geq 1$

and every $x \in [0, 2m]$

$$g_t(x) \leq \min\left(\sqrt{x}, \sqrt{2m-x}\right)\left(1 - \frac{\phi^2}{2}\right)^t + \frac{x}{2m}$$

LS[Thm]: For every distrib p^0

For every t

we have

① If $x \le m$ then

$$g_t(x) \le \frac{g_{t-1}(x - 2\phi x) + g_{t-1}(x + 2\phi}{2}$$

Not

$g_{t-1}\left(\frac{x}{2} - \phi x\right)$

② If $x > m$ then

$$g_t(x) \le \frac{g_{t-1}(x - 2\phi(2m - x)) + g_{t-1}(\ )}{2}$$

Proof: [of item 1]

Will show this for hige point x $\in$ set S

$$g_t(x) = \sum_{i=1}^{x} p^t(e_i)$$

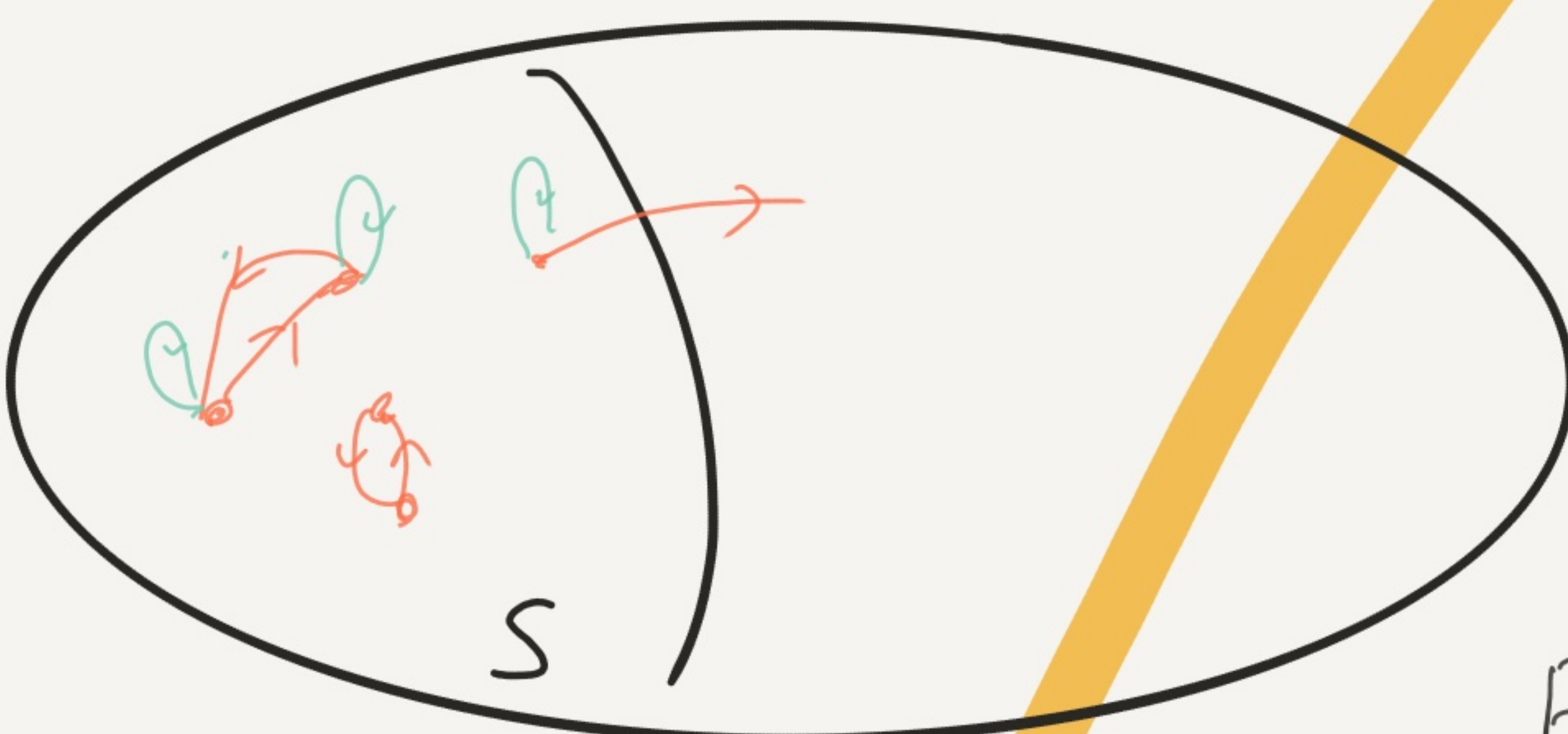
$$= \sum_{i=1}^{x} p^t(u_i, v_i) = \sum_{u \in S} p^t(u)$$

$$g_t(x) = \sum_{u \in S} p^t(u)$$

$$= \sum_{i=1}^{x} p^{t-1}(v_i, u_i) = \sum_{E_1} p^{t-1}(v_i, u_i) + \sum_{E_2} p^{t-1}(v_i, u_i)$$

E_1 = edges internal to S [no loops]

$E_2 = \{(v_i, u_i) : u_i \in S, v_i \notin S\}$ and loops



$$|E_1| \leq x - \phi x - \frac{x}{2}$$

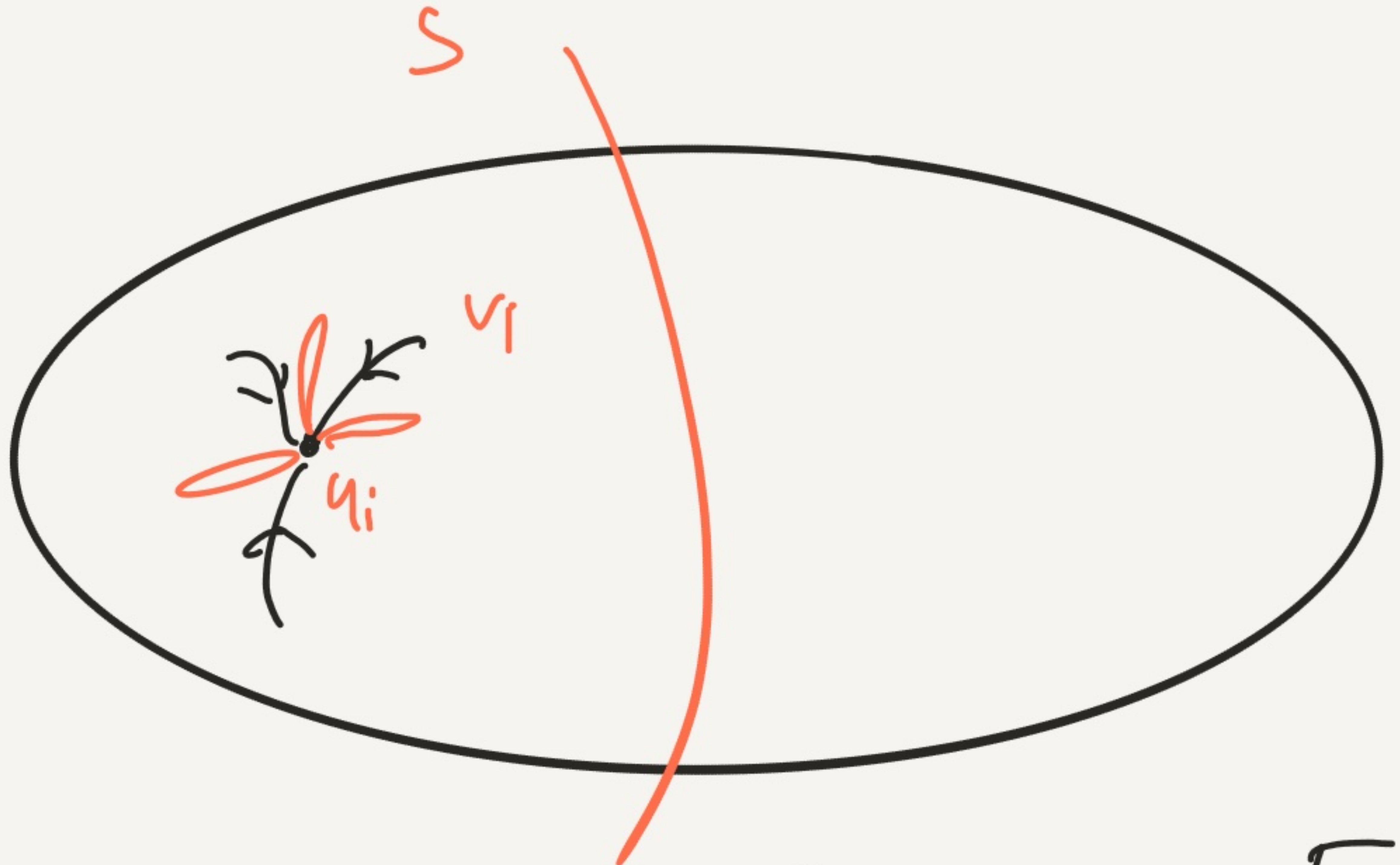
$$\Rightarrow |E_1| \leq \frac{x}{2} - \phi x$$

Notice

$$\leq g_{t-1}\left(\frac{x}{2} - \phi x\right)$$

We will define another set of edges which we denote as E_1'

$E_1' = E_1 \uplus$ Attach a unique loop to each edge in S_1



For $e_i \in E_1'$ you have

$$a_{e_i} = \begin{cases} \frac{1}{2} & \text{if } e_i \in E_1 \\ \frac{1}{2} & \text{if } e_i = e_i^* \end{cases}$$

$$|E_1'| \leq 2|E_1| \leq 2\left(\frac{x}{2} - \phi x\right)$$

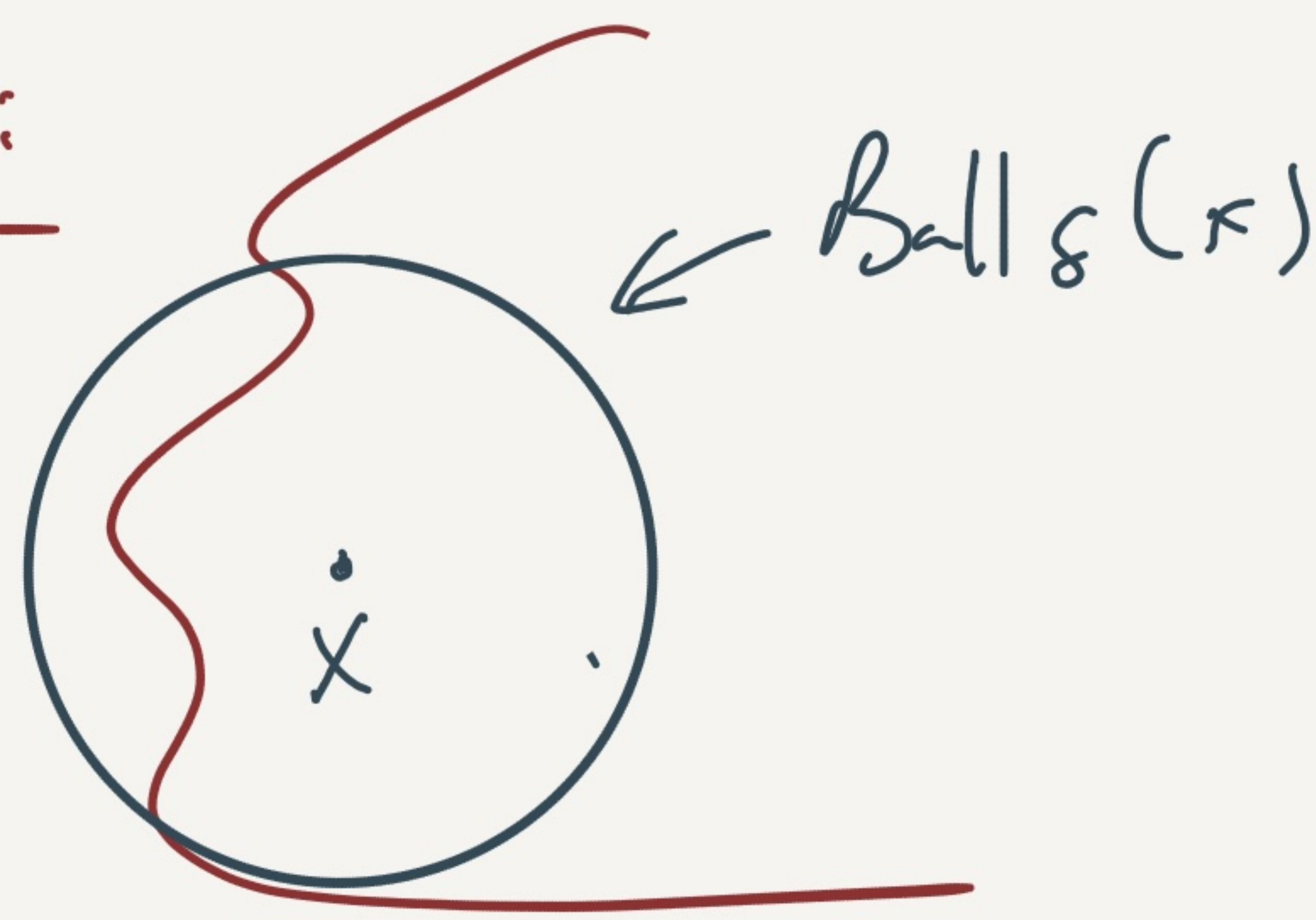
$$= x - 2\phi x$$

$$\sum_{E_1} \rho^{t-1}(v_i, u_i) = \frac{1}{2} \sum_{E_1'} \rho^{t-1}(e_i)$$

$$\leq \frac{1}{2}\, g_{t-1}(x - 2\phi x)$$

Similarly $\dfrac{g_{t-1}(x+2\phi x)}{2}$ also follows

<u>Ball Walk :</u>



$X_t = x$

Pick $y \sim \text{Ball}_\delta(x)$

If $y \in K$

$X_{t+1} = y$

o/w

$X_{t+1} = x$

$\mu \equiv \text{Unif}[K]$

Greedy aggregation rule:

For $x \in [0,1]$

$$g_t(x) = \sup_{\substack{A \in \mathcal{B}(K) \\ \mu(A) = x}} p^t(A)$$

Alternately

$$g_t(x) = \sup_{z \in K} \int c(z) \cdot p^t(z)$$

st

$$c : K \to [0,1]$$

$$\int_{z \in K} c(z)\, \mu(z) = x$$

In graph case

$$g_t(x) \leq \frac{g_{t-1}(x - 2\phi x) + g_{t-1}(x + 2\phi x)}{2}$$

For $A \in \mathcal{B}(K)$

$$\phi(A) = \frac{\Phi(A)}{\min\{\mu(A), \mu(\bar{A})\}}$$

Ergodic flow across A

$$= \sum_{\substack{x \in A \\ y \in \bar{A}}} \mu(x) \cdot P(x, y)$$

$$= \int_{z \in A} P_z(K \setminus A) \cdot d\mu(z)$$

Lemma: let P be an "atom free distsb"

For every $0 \le x \le \frac{1}{2}$

$$g_t(x) \le \frac{g_{t-1}(x - 2\phi x) + g_{t-1}(x + 2\phi x)}{2}$$

LS [Thm 2]:

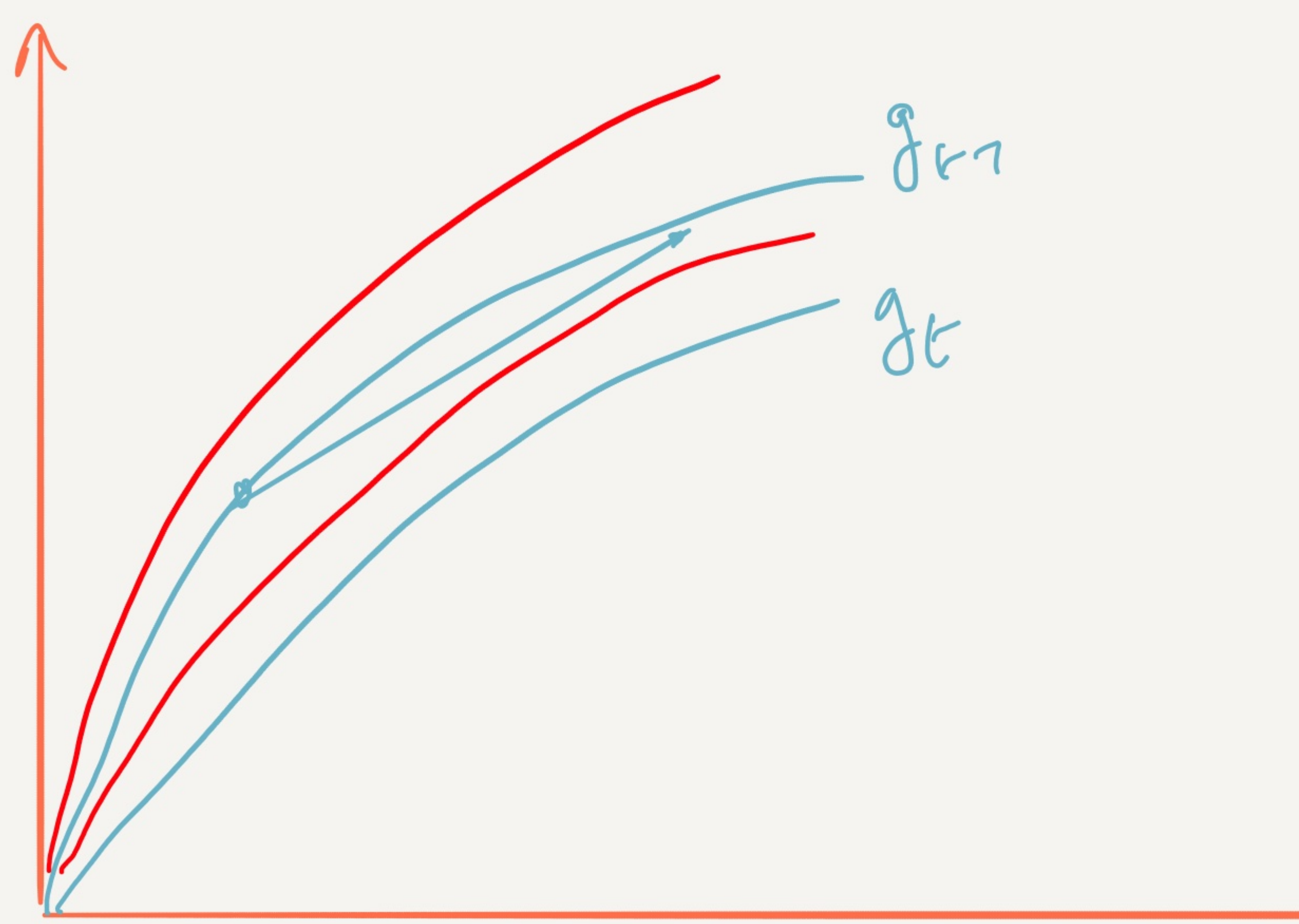
For every distrib $\vec{p}_0$

every $t \geq 1$

and every $x \in [0, 2m]$

$$g_t(x) \leq \min\left(\sqrt{x}, \sqrt{2m-x}\right)\left(1 - \frac{\phi^2}{2}\right)^t + \frac{x}{2m}$$

Proof:



$$R^t(x) = \min\left(\sqrt{x}, \sqrt{2m-x}\right)\left(1-\frac{\phi^2}{2}\right)^t + \frac{x}{2m}$$

$$R^0(x) = \min\left(\sqrt{x}, \sqrt{2m-x}\right) + \frac{x}{2m}$$

<u>Claim:</u> $g_0(x) \leq R_0(x)$

$g_0 \equiv g_{\mathbb{1}_v} \leq R_0$

Assume we showed

$$R_t(x) \leq \frac{1}{2}\left[R_{t-1}(x-2\phi x) + R_{t-1}(x-2\phi x)\right]$$

$$= \frac{1}{2}\left[\sqrt{x-2\phi x} + \sqrt{x+2\phi x}\right] + \frac{x}{2m}$$

$$\leq \sqrt{x}\left(1-\frac{\phi^2}{2}\right) + \frac{x}{2m}$$