Lecture 8

Sunday, March 28, 2021 10:01 AM

Cutting plane methods

Many combinatorial optimization problems which can be cast as linear programs but with exponentially many constraints.

Example 1: Minimum spanning tree problem

Given a graph $G = (V, E)$, cost function $c : E \to \mathbb{R}$, find a spanning tree with minimum cost.

Integer programming formulation

Let x_e denote the indicator variable for whether the edge e is present in the supposed spanning tree

$$\min_{x} \quad \langle c, x \rangle$$

$$\text{s.t.} \quad \sum_{e \in E} x_e = n - 1$$

$$\sum_{e \in E(S,S)} x_e \leq |S| - 1 \quad \forall \ \phi \subsetneq S \subsetneq V$$

$$x_e \in \{0, 1\}$$

Linear programming relation: Relax $x_e \in \{0,1\}$ to $0 \leq x_e \leq 1$

Linear programming relaxation is tight!

Example 2: Maximum weight perfect matching

Given a graph $G = (V, E)$, a weight function $w : E \to \mathbb{R}$, find a perfect matching with maximum weight.

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Integer programming formulation

Let x_e denote the indicator variable for whether edge e is present in the supposed perfect matching

$$\max_x \ \langle w, x \rangle$$

$$\text{s.t.} \quad \sum_{e \in \delta(u)} x_e = 1 \quad \forall \ u \in V$$

$$\sum_{e \in E(S,S)} x_e \leq \frac{|S|-1}{2} \quad \forall \ \emptyset \subsetneq S \subsetneq V \text{ s.t. } |S| \text{ odd}$$

$$x_e \in \{0,1\}$$

Linear programming formulation: Relax $x_e \in \{0,1\}$ to $0 \leq x_e \leq 1 \ \forall \ e \in E$

Again the linear programming relaxation is tight.

Can we use interior point methods? Currently no. It is not clear how to design barrier functions for these polytopes which have efficiently computable gradients, Hessians & also small complexity parameter.

The above LPs have exponentially many constraints but we can design an efficient separation oracle.

Definition [Separation oracle]: A separation oracle for a convex set $K \subseteq \mathbb{R}^n$ is an oracle of the following kind:

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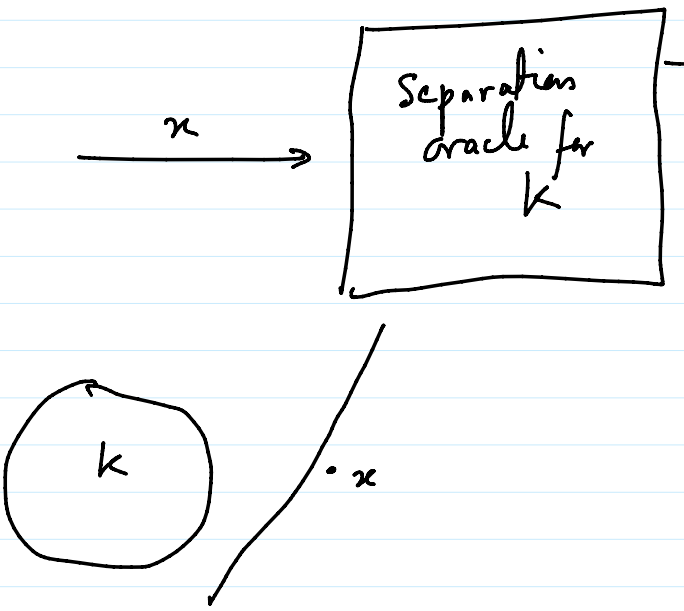

- Applications & some separation oracles in the next lecture

[Khachiyan 79]: Designed the first polynomial time algorithm for linear programming (poly many constraints) using the Ellipsoid algorithm (most famous cutting plane method).

[Karmarkar 84]: Poly time algorithm for linear programming using interior point methods.

Cutting plane methods don't work so well in practice. But interior point methods work well in practice.

To solve the LP $\min_{x \in P} \langle c, x \rangle$

↓

Can reduce it a feasibility problem: find a point in $P \cap \{x : \langle c, x \rangle \leq \gamma\}$

& binary search over γ.

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& binary search over γ.

So a more general problem: Given a convex set $K \subseteq \mathbb{R}^n$ described by a separation oracle, find a point inside K.

Assumptions: $\exists\ y$ s.t. $B(y,r) \subseteq K \subseteq B(0,R)$

(y: unknown)

$B(y,r) = \{x \in \mathbb{R}^n : \|x-y\|_2 \leq r\}$

Cutting plane meta algorithm

Input: A separation oracle for the set K

Output: $\hat{x} \in K$

Algorithm: start with $E_0 = B(0,R)$

while $\mathrm{vol}(E_t) \geq \mathrm{vol}(B(y,r))$

select a point $x_t \in E_t$

query the separation oracle for K at x_t.

if $x_t \in K$, then output $\hat{x} = x_t$

else

let $H_t = \{x : \langle x, h_t \rangle \leq \langle x_t, h_t \rangle\}$

be the hyperplane output by the oracle

construct E_{t+1} s.t.

$E_t \cap H_t \subseteq E_{t+1}$

end

Proposition [Number of iterations of the meta algorithm]:

Suppose there is a volume drop at each step i.e.

$\text{vol}(E_{t+1}) \leq \alpha \, \text{vol}(E_t) \quad \forall \, t$, for some $0 < \alpha < 1$.

Then after $O\left(\frac{n \log\left(\frac{R}{r}\right)}{\log\left(\frac{1}{\alpha}\right)}\right)$ iterations, the algorithm outputs a point $\hat{x} \in K$.

Proof: Note that $K \subseteq E_t \quad \forall \, t$. Because of the separating hyperplane property.

$$\text{vol}(E_0) = C_n R^n$$

$$\text{vol}(E_T) \geq \text{vol}(K) \geq C_n r^n$$

$$\Rightarrow \frac{\text{vol}(E_T)}{\text{vol}(E_0)} \geq \left(\frac{r}{R}\right)^n$$

also $\text{vol}(E_T)/\text{vol}(E_0) \leq \alpha^T$

$$\Rightarrow \alpha^T \geq \left(\frac{r}{R}\right)^n$$

$$\Rightarrow T \leq \frac{n \log\left(\frac{R}{r}\right)}{\log\left(\frac{1}{\alpha}\right)}$$

How to choose E_t & x_t to ensure a volume drop?

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What if

$$E_{t+1} := E_t \cap H_t \; ?$$

x_t needs to be some kind of center of G_t.

Centre of gravity of a convex set E, $c := \frac{1}{vol(E)} \int_{x \in E} x \, dx$

[Grünbaum 60]: For any convex set $E \subseteq \mathbb{R}^n$ with centre of gravity $c \in \mathbb{R}^n$, & any halfspace $H = \{x : \langle h, x \rangle \leq \langle h, c \rangle\}$ it holds that

$$\frac{1}{e} \leq \frac{vol(K \cap H)}{vol(K)} \leq 1 - \frac{1}{e}$$

But centre of gravity is NP-hard to compute!!

This brings us to Khachiyan's Ellipsoid algorithm

The sets E_t's will be Ellipsoids

$$E(x, M) := \left\{ y : \|y - x\|_{M^{-1}}^2 \leq 1 \right\}$$

$$\downarrow$$

$$\langle y - x, M^{-1}(y - x) \rangle$$

$x \in \mathbb{R}^n$, M pd $n \times n$ matrix

$E_0 = B(0, R)$

— — min volume Ellipsoid enclosing

$E_0 = B(0, R)$

$E_{t+1} :=$ min volume Ellipsoid enclosing $E_t \cap H_t$.

Lemma (Volume drop): Let $E(x_0, M) \subseteq \mathbb{R}^n$ be an Ellipsoid. Then $\exists$ an Ellipsoid $E(x', M')$ s.t.

$$E(x_0, M) \cap \{x : \langle x, h \rangle \leq \langle x_0, h \rangle\} \subseteq E(x', M')$$

$$\& \quad vol(E(x', M')) \leq e^{-\frac{1}{2(n+1)}} vol(E(x_0, M))$$

Moreover x' & M' are explicitly described as follows:

$$x' := x_0 - \frac{1}{n+1} Mg$$

$$M' := \frac{n^2}{n^2-1} \left(M - \frac{2}{n+1} Mg (Mg)^t \right)$$

where $g := \frac{h}{\|h\|_M}$

The lemma gives $O\left(n^2 \log\left(\frac{R}{r}\right)\right)$-time cutting plane method.

Exercise: $E(x', M')$ is also the minimum volume Ellipsoid enclosing $E(x_0, M) \cap \{x : \langle h, x \rangle \leq \langle h, x_0 \rangle\}$.