Sampling Bases of a Matroid

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- **Contraction:** Let $M = ([n], \mathcal{I})$ be a matroid and $S \in \mathcal{I}$. Then the contraction M/S is the matroid with ground set $[n] \setminus S$ and independent sets $\{T \subseteq [n] \setminus S \mid T \cup S \in \mathcal{I}\}$.

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Example: Consider the matroid $M = ([4], \{\varphi, \{1\}, \{2\}, \{3\}, \{1,3\}, \{2,3\}\})$. Then, $\text{rank}([4]) = 2$, the bases are $\{1,3\}$ and $\{2,3\}$. 4 is a loop. $\{1,2\}$ is a parallel. For $S = \{3\}$, $M' = M/S = (\{1,2,4\}, \{\varphi, \{1\}, \{2\}\})$ is a contraction.

Problem

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We will see the polynomial time randomised approximation algorithm proposed in "**Log-Concave Polynomials II: High-Dimensional Walks and an FPRAS for Counting Bases of a Matroid.** Nima Anari, Kuikui Liu, Shayan Oveis Gharan, and Cynthia Vinzant" that appeared in STOC 2019.

- FPRAS to sample a uniform random basis of a matroid $=>$ FPRAS to count the number of bases of a matroid (Since Sampling $\leftrightarrow$ Counting).
- Expansion of the bases exchange graph of a matroid is 1.

Outline

- Preliminaries
  - Linear Algebra
  - Simplicial Complexes
- Walks on Simplicial Complexes and some known results
- Strongly log-concave polynomials
- Main results of the paper
- Some applications

Linear Algebra

A matrix $A \in \mathbb{R}^{n\times n}$ is stochastic if all $A_{ij} \geq 0$ and $\sum_{j\in[n]} A_{ij} = 1, \forall i \in [n]$.

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Cauchy's Interlacing Theorem [HJ13][i]*:* For a symmetric matrix $A \in \mathbb{R}^{n \times n}$ and vector $v \in \mathbb{R}^n$, the eigenvalues of A interlace the eigenvalues of $A + vv^\top$. That is, for $B = A + vv^\top$,

$$\lambda_n(A) \leq \lambda_n(B) \leq \lambda_{n-1}(A) \leq \cdots \leq \lambda_2(B) \leq \lambda_1(A) \leq \lambda_1(B).$$

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Lemma: Let $A \in \mathbb{R}^{n \times n}$ be a symmetric matrix and let $P \in \mathbb{R}^{m \times n}$. If A has at most one positive eigenvalue, then $PAP^\top$ has at most one positive eigenvalue.

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Proof: Given A has at most one positive eigenvalue. Therefore, write $A = B + vv^\top$ for some vector $v \in \mathbb{R}^n$ and for some $B \preccurlyeq 0$. Then $PAP^\top = PBP^\top + Pvv^\top P^\top$.
Then,

$$x^\top PBP^\top x = (P^\top x)^\top B(P^\top x) \leq 0; \qquad \forall x \in \mathbb{R}^m.$$

Therefore, $PBP^\top \preccurlyeq 0$.

Lemma: Let $A \in \mathbb{R}^{n \times n}$ be a symmetric matrix and let $P \in \mathbb{R}^{m \times n}$. If A has at most one positive eigenvalue, then $PAP^\top$ has at most one positive eigenvalue.

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Therefore, $PBP^\top \preccurlyeq 0$.

Let $w = Pv \in \mathbb{R}^m$. Then $Pvv^\top P^\top = ww^\top$. By the Cauchy interlacing theorem,

$$\lambda_2\left(PBP^\top + (Pv)(Pv)^\top\right) \leq \lambda_1\left(PBP^\top\right) \leq \lambda_1\left(PBP^\top + (Pv)(Pv)^\top\right),$$

Since all eigenvalues of $PBP^\top$ are nonpositive, $PAP^\top = PBP^\top + ww^\top$ has at most one positive eigenvalue.

$\square$

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Proof: Since $B \succcurlyeq 0$, we can write $B = C^\top C$ for some $C \in \mathbb{R}^{n \times n}$. By the fact above, $BA = C^\top CA$ hasthe same nonzero eigenvalues as the matrix $CAC^\top$. Since A has at most one positive eigenvalue, by the previous lemma, $CAC^\top$ has at most one positive eigenvalue and so does BA.

$\square$

Lemma: Let $A \in \mathbb{R}^{n \times n}$ be a symmetric matrix with nonnegative entries and at most one positive eigenvalue, and let $w(i) = \sum_{j=1}^{n} A_{i,j}$. Then,

$$A \preccurlyeq \frac{ww^\top}{\sum_i w(i)}.$$

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Proof: Let $W = \mathrm{diag}(w)$. Then, $\mathcal{A} = \underbrace{W^{-1/2}AW^{-1/2}}_{PAP^\top}$ has at most one positive eigenvalue. Observe that the top eigenvector of $\mathcal{A}$ is the $\sqrt{w}$ vector, where $\sqrt{w}(i) = \sqrt{w(i)}$, for all i. In particular, $\mathcal{A}\sqrt{w} = \sqrt{w}$. Therefore, $\sqrt{w}$ is the only eigenvector of $\mathcal{A}$ with positive eigenvalue and we have

$$\mathcal{A} \preccurlyeq \frac{\sqrt{w}\sqrt{w}^\top}{\|\sqrt{w}\|^2} \preccurlyeq \frac{\sqrt{w}\sqrt{w}^\top}{\sum_i w(i)}$$

Multiplying both sides of the inequality on the left and right by $W^{1/2}$ proves the lemma.

Theorem ((Courant-Fischer Theorem): . Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear operator that is self-adjoint with respect to some inner product $\langle \cdot, \cdot \rangle$ (not necessarily Euclidean). If $\lambda_n \leq \cdots \leq \lambda_1$ are the eigenvalues of T, then,

$$\lambda_k = \min_{\mathcal{U}} \quad \max_{v} \langle v, Tv \rangle,$$

where the minimum is taken over all $(n-k)$-dimensional subspaces $\mathcal{U} \subseteq \mathbb{R}^n$ and the maximum is taken over all the vectors $v \in \mathcal{U}$ such that $\langle v, v \rangle = 1$.

Markov Chains

A Markov chain is a triple (Ω, P, π),

- Ω denotes the finite state space.
- $P \in \mathbb{R}_{\geq 0}^{\Omega \times \Omega}$ denotes the transition probability matrix.
- $\pi \in \mathbb{R}_{\geq 0}^{\Omega}$ denotes the stationary distribution of the chain.

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A chain (Ω, P, π) is reversible if there is a nonzero nonnegative function $f : \Omega \to \mathbb{R}_{\geq 0}$ such that for any pair of states $\tau, \sigma \in \Omega$,

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Reversible Markov chain can be realized as random walks on weighted graphs $G = (V, E, w)$,

- From a vertex u, choose a neighbour v with probability proportional $w(\{u, v\})$.
- Then, $\pi(u) \propto w(u) = \sum_{v:\{u,v\} \in E} w(\{u, v\})$.
- Its an ϵ-lazy random walk when we stay at the vertex with probability ϵ.

Reversible Markov Chains

If a Markov chain is reversible then,

1. f is propotional to the stationary distribution π.

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2. P is self-adjoint w.r.t. the following $\langle \cdot, \cdot \rangle$ defined for $\varphi, \psi \in \mathbb{R}^{\Omega}$:

$$\langle \varphi, \psi \rangle = \sum_{x \in \Omega} f(x) \varphi(x) \psi(x).$$

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3. the largest eigenvalue of P is 1 (By point 2 and Courant-Fischer Theorem).

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3. the largest eigenvalue of P is 1 (By point 2 and Courant-Fischer Theorem).
4. Let $\lambda^* = \max\{|\lambda_2|, |\lambda_n|\}$. Then we have the following bound on mixing time, *Theorem [DS91, Prop 3]*[ii] *:* For any $\epsilon > 0$ and any $\tau \in \Omega$,

$$t_\tau(\epsilon) \leq \frac{1}{1 - \lambda^*(P)} \cdot \log\left(\frac{1}{\epsilon \cdot \pi(\tau)}\right).$$

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Cheeger's Inequality

Definition (Conductance): Conductance of $G = (V, E, w)$ is,

$$\text{cond}(G) = \min_{\varphi \subsetneq S \subsetneq V} \text{cond}(S) = \frac{w(E(S, \overline{S}))}{\text{vol}(S)} = \frac{\sum_{e \in E(S,\overline{S})} w(e)}{\sum_{u \in S} w(u)}.$$

[iii] N. Alon and V. Milman. "Isoperimetric inequalities for graphs, and superconcentrators". In: Journal of Combinatorial Theory, Series B 38.1 (Feb. 1985), pp. 73–88.

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Theorem (Cheeger's Inequlity) [AM85[iii], Alo86[iv]]: For any d-regular weighted graph $G = (V, E, w)$,

$$\frac{d - \lambda_2(A_G)}{2} \leq \mathrm{cond}(G) \leq \sqrt{2(d - \lambda_2(A_G))}.$$

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For any $1 \leq k \leq n$, we define the set of k-faces/k-simplices as,

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We say that X is **pure** of dimension d if all maximal faces of X have dimension d.

Simplicial Complexes: Link of a face

The link of a face $\tau \in X$ denoted by X_τ is the simplicial complex on $[n] \setminus \tau$ obtained by taking all faces in X that contain τ and removing τ from them,

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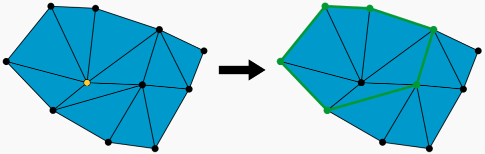

Figure: A simplex on the ground set of 11 vertices. $X(1)$ is the set of all vertices. $X(2)$ is the set of all edges. $X(3)$ is the set of all blue triangles. $X(4) = \cdots = X(11) = \Phi$. Yellow vertex's link is the set of green edges. Source: Wikipedia.

Simplicial Complexes: Weight of a face

A weight function $w : X \to \mathbb{R}_{>0}$, which assigns a positive weight to each face of X, is balanced if for every non-maximal face $\tau \in X$ of dimension k,

$$w(\tau) = \sum_{\sigma \in X(k+1) : \sigma \supset \tau} w(\sigma).$$

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For a pure simplicial complex of dimension d, we can define a balanced weight function such that for any $\tau \in X(k)$,

$$w(\tau) = (d-k)! \sum_{\sigma \in X(d) : \sigma \supset \tau} w(\sigma),$$

where the weights of the d-faces are arbitrarily assigned.

Simplicial Complexes: 1-skeleton of X

The 1-skeleton of X is the graph $G\left(X(1), X(2)\right)$.

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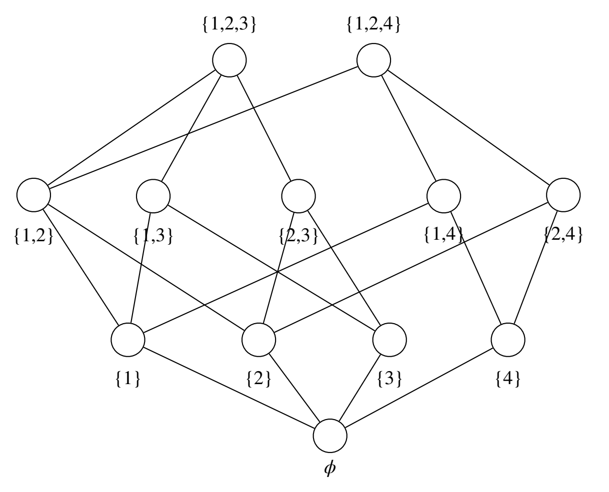
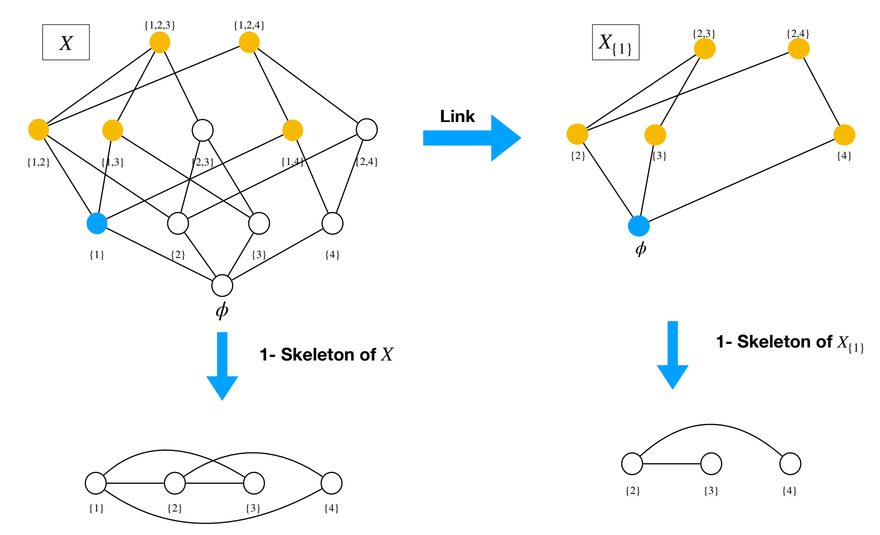
We will also use 1-skeleton of link of a face τ, that is the graph $G\left(X_\tau(1), X_\tau(2)\right)$. Recall,

$$X_\tau = \{\sigma : \sigma \cup \tau \in X\}.$$

Simplicial Complex: Example




X
{1,2,3}
{1,2,4}
{1,2}
{1,3}
{2,3}
{1,4}
{2,4}
{1}
{2}
{3}
{4}
ϕ
Link
$X_{\{1\}}$
{2,3}
{2,4}
{2}
{3}
{4}
ϕ
1- Skeleton of X
1- Skeleton of $X_{\{1\}}$
{1}
{2}
{3}
{4}
{2}
{3}
{4}

Matroids as Simplicial Complexes

For any matroid $M = ([n], \mathcal{I})$ of rank r, the independent sets $\mathcal{I}$ form a pure r-dimensional simplicial complex on $[n]$ called its independence (or matroid) complex.

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Example: Consider again the matroid $M = ([4], \{\varphi, \{1\}, \{2\}, \{3\}, \{1,3\}, \{2,3\}\})$. Then, $\mathrm{rank}([4]) = 2$, the bases are $\{1,3\}$ and $\{2,3\}$. 4 is a loop. $\{1,2\}$ is a parallel. For $S = \{3\}$, $M' = M/S = (\{1,2,4\}, \{\varphi, \{1\}, \{2\}\})$ is a contraction.

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The corresponding simplicial complex is X such that $X(0) = \{\varphi\}$, $X(1) = \{\{1\}, \{2\}, \{3\}\}$, $X(2) = \{\{1,3\}, \{2,3\}\}$. This is a pure 2-dimensional simplicial complex. For $\tau = \{3\}$, its link X_τ is the simplicial complex with faces $X_\tau(0) = \{\varphi\}, X_\tau(1) = \{\{1\}, \{2\}\}$.

Outline

- Preliminaries
  - Linear Algebra
  - Simplicial Complexes
- Walks on Simplicial Complexes and some known results
- Strongly log-concave polynomials
- Main results of the paper
- Some applications

Walks on Simplicial Complexes

Define weighted complex (X, w) as a pure d-dimensional simiplicial complex with a balanced weight function w. Then, in a graph representation,

- Let G_k represent a bipartite graph with $X(k)$ and $X(k+1)$ as the two partitions.

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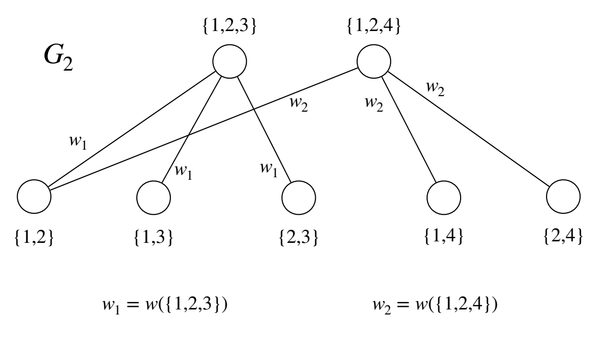
- Let G_k represent a bipartite graph with $X(k)$ and $X(k+1)$ as the two partitions.
- (τ, σ) forms an edge iff $\tau \subset \sigma$, and its weight is $w(\sigma)$.
- Now define two simple (weighted) random walks on G_k, one of $X(k)$ called $P_k^{\wedge}$ and the other on $X(k+1)$ called $P_{k+1}^{\vee}$.



Walks on Simplicial Complexes

- **Upper k-walk:** A movement from a face τ in $X(k)$ to a higher dimensional face and back to a face τ' in $X(k)$.

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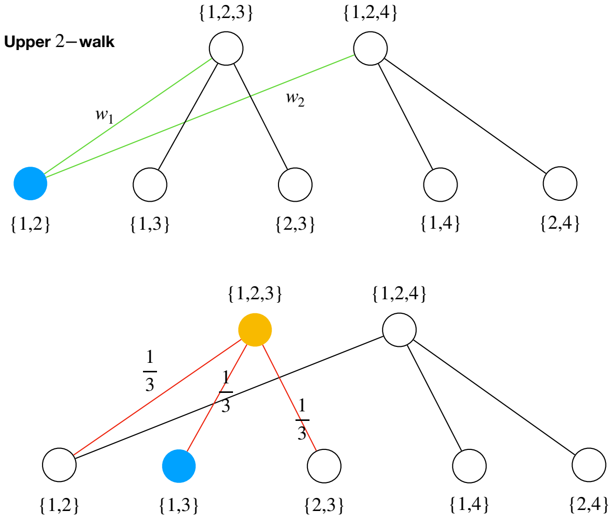
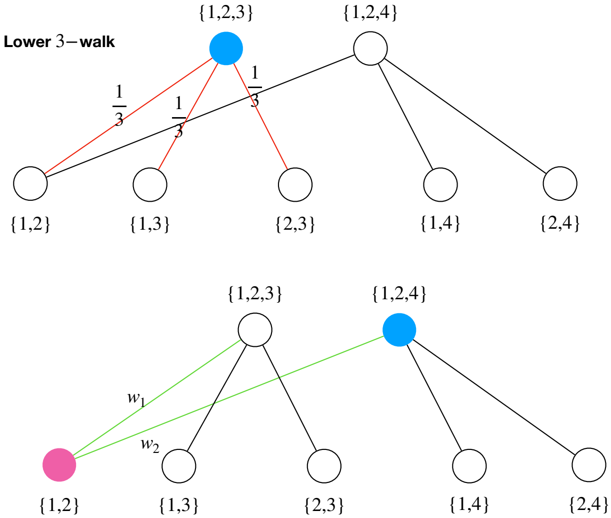
- **Lower k-walk:** A movement from a face σ in $X(k+1)$ to a lower dimensional face and back to a face σ' in $X(k)$. This is given by the transition matrix,

$$P_{k+1}^{\vee}(\sigma,\sigma') = \begin{cases} \sum_{\tau\in X(k):\tau\subset\sigma} \frac{w(\sigma)}{(k+1)w(\tau)} & \text{if } \sigma = \sigma' \\ \frac{w(\sigma')}{(k+1)w(\sigma\cap\sigma')} & \text{if } \sigma\cap\sigma' \in X(k) \\ 0 & \text{otherwise} \end{cases}.$$


Upper 2−walk
{1,2,3}
{1,2,4}
w_1
w_2
{1,2}
{1,3}
{2,3}
{1,4}
{2,4}
{1,2,3}
{1,2,4}
$\frac{1}{3}$
$\frac{1}{3}$
$\frac{1}{3}$
{1,2}
{1,3}
{2,3}
{1,4}
{2,4}


Lower $3-$walk
{1,2,3}
{1,2,4}
$\frac{1}{3}$
$\frac{1}{3}$
$\frac{1}{3}$
{1,2}
{1,3}
{2,3}
{1,4}
{2,4}
{1,2,3}
{1,2,4}
w_1
w_2
{1,2}
{1,3}
{2,3}
{1,4}
{2,4}

Walks on Simplicial Complexes

Both the random walks using the transition matrices $P_k^{\wedge}$ and $P_{k+1}^{\vee}$ are reversible w.r.t. w, i.e., for any $\tau, \tau' \in X(k)$,

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Therefore, upper k-walk and lower $(k-1)$-walk have the same stationary distribution π_k, such that $\forall \tau \in X(k)$, $\pi_k(\tau) \propto w(\tau)$.

Lemma 1: For any $1 \leq k < d$, $P_k^\wedge$ and $P_{k+1}^\vee$ are stochastic, self-adjoint w.r.t. the w-induced inner product, PSD, and have the same (with multiplicity) non-zero eigenvalues.

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Proof: Let P_k be the transition matrix of a simple random walk on G_k,

$$P_k = \begin{bmatrix} \mathbf{0} & P_k^{\downarrow} \\ P_k^{\uparrow} & \mathbf{0} \end{bmatrix} \implies P_k^2 = \begin{bmatrix} P_k^{\downarrow} P_k^{\uparrow} & \mathbf{0} \\ \mathbf{0} & P_k^{\uparrow} P_k^{\downarrow} \end{bmatrix} \implies P_k^2 = \begin{bmatrix} P_{k+1}^{\vee} & \mathbf{0} \\ \mathbf{0} & P_k^{\wedge} \end{bmatrix},$$

where $P_k^{\downarrow} \in \mathbb{R}^{X(k+1) \times X(k)}$ and $P_k^{\uparrow} \in \mathbb{R}^{X(k) \times X(k+1)}$ are stochastic matrices.

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$\therefore$ Both $P_k^{\wedge}$ and $P_{k+1}^{\vee}$ are self-adjoint w.r.t. the w-induced inner product, are PSD, and stochastic, and have the same eigenvalues by the fact that AB and BA have same nonzero eigenvalues.

$\square$

Local Spectral Expanders

$P_1^{\wedge}$ is the transition probability matrix of the simple $\frac{1}{2}$-lazy random walk on the weighted 1-skeleton of X. Then the non-lazy transition matrix is,

$$\tilde{P}_1^{\wedge} = 2\left(P_1^{\wedge} - \frac{I}{2}\right).$$

[v] Tali Kaufman and Izhar Oppenheim. "High Order Random Walks: Beyond Spectral Gap". In: APPROX/RANDOM. 2018, 47:1–47:17.

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Definition (Local Spectral Expanders) [KO18][v]: For $\lambda \geq 0$, a pure d-dimensional weighted complex (X, w) is a λ-local-spectral-expander if for every $0 \leq k < d-1$, and for every $\tau \in X(k)$, we have $\lambda_2\left(\tilde{P}_{\tau,1}^{\wedge}\right) \leq \lambda$.

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Walks on Simplicial Complexes

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Proof: Let $M = P_k^\wedge - \left(\frac{k}{k+1} P_k^\vee + \frac{1}{k+1} I\right)$.
Fix $\eta \in X(k-1)$ and define the matrix M_η with the entries as,

$$M_\eta(\tau, \sigma) = \begin{cases} M(\tau, \sigma) & \text{if } \tau \neq \sigma, \tau \cap \sigma = \eta \\ -\frac{1}{k+1} \cdot \frac{w(\tau)}{w(\eta)} & \text{if } \tau = \sigma, \tau \supset \eta \\ 0 & \text{otherwise} \end{cases}.$$

Note that $M = \sum_{\eta \in X(k-1)} M_\eta$. Hence, enough to show $M_\eta \preccurlyeq 0, \forall \eta \in X(k-1)$.

Fix $\eta \in X(k-1)$. We can write M_η as

$$M_\eta = \frac{1}{(k+1)w(\eta)} \operatorname{diag}(w_\eta)^{-1} \cdot \left(w(\eta) \cdot A_\eta - w_\eta w_\eta^\top \right),$$

where w_η is the $|X(k)|$-dimensional vector whose non-zero entries are $w(\tau)$ for $\tau \supset \eta$, and A_η is the $|X(k)| \times |X(k)|$ matrix whose non-zero entries are $w(\tau \cup \sigma)$ for $\tau, \sigma \in X(k)$ satisfying $\tau \cup \sigma \in X(k+1)$ and $\tau \cap \sigma = \eta$.

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$$\langle v, M_\eta v \rangle = v^\top \operatorname{diag}\left(w_k\right) M_\eta v = v^\top \operatorname{diag}\left(w_\eta\right) M_\eta v,$$

where w_k is the vector of w values on $X(k)$ and for the last equality we used that w_k is the same as w_η on all $\tau \supset \eta$.

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where w_k is the vector of w values on $X(k)$ and for the last equality we used that w_k is the same as w_η on all $\tau \supset \eta$.

Therefore, to show that M_η is NSD w.r.t. the inner product induced by w, it is enough to show that $\operatorname{diag}(w_\eta) M_\eta$ is NSD in the usual sense, i.e., show that $A_\eta \preccurlyeq \frac{w_\eta w_\eta^\top}{w(\eta)}$.

A_η is the weighted adjacency matrix of the 1-skeleton (which we recall is a graph) of the link X_η. Then $\tilde{P}^{\wedge}_{\eta,1} = \frac{1}{k+1}\mathrm{diag}(w_\eta)^{-1}A_\eta$ gives its non-lazy simple random walk matrix.

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(X, w) is a 0-local spectral expander $\implies$ $\tilde{P}^{\wedge}_{\eta,1}$ has at most one positive eigenvalue, whence $A_\eta = (k+1)\mathrm{diag}(w_\eta) \cdot \tilde{P}^{\wedge}_{\eta,1}$ has at most one positive eigenvalue by Lemma (If $A \in \mathbb{R}^{n\times n}$ be a symmetric matrix with at most one positive eigenvalue. Then, for any PSD matrix $B \in \mathbb{R}^{n\times n}$, BA has at most one positive eigenvalue).

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We know that the weights are balanced. Therefore from Lemma (If $A \in \mathbb{R}^{n\times n}$ be a symmetric matrix with nonnegative entries and at most one positive eigenvalue, and $w(i) = \sum_{j=1}^{n} A_{i,j}$. Then, $A \preccurlyeq \frac{ww^\top}{\sum_i w(i)}$) it follows that $A_\eta \preccurlyeq \frac{w_\eta w_\eta^\top}{w(\eta)}$.

Local Spectral Expanders

Theorem 1 [KO18][vi]*:* Let (X, w) be a pure d-dimensional weighted 0-local spectral expander and let $0 \leq k < d$. Then, for all $-1 \leq i \leq k$, $P_k^{\wedge}$ has at most $|X(i)| \leq \binom{n}{i}$ eigenvalues of value $> 1 - \frac{i+1}{k+1}$, where for convenience, we set $X(-1) = \varphi$ and $\binom{n}{-1} = 0$. In particular, the second largest eigenvalue of $P_k^{\wedge}$ is at most $\frac{k}{k+1}$.

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Induction step: Assume the claim holds for all $0 \leq k < d - 1$. Then,

$$P_{k+1}^{\wedge} \preccurlyeq \frac{k+1}{k+2} P_{k+1}^{\vee} + \frac{1}{k+2} I \qquad \text{. . . from Lemma 2.}$$

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For $-1 \leq i \leq k$, $P_k^{\wedge}$ has at most $|X(i)|$ eigenvalues $> 1 - \frac{i+1}{k+1}$, by the induction hypothesis.

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Hence, $P^{\wedge}_{k+1}$ has at most $|X(i)|$ eigenvalues $> \frac{k+1}{k+2}\left(1 - \frac{i+1}{k+1}\right) + \frac{1}{k+1} = 1 - \frac{i+1}{k+2}$.

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$P_{k+1}^{\wedge} \in \mathbb{R}^{|X(k+1)| \times |X(k+1)|}$. Therefore, for $i = k+1$, $P_{k+1}^{\wedge}$ has at most $|X(k+1)|$ eigenvalues > 0.

$\square$

Outline: Thursday

- Preliminaries
  - Linear Algebra
  - Simplicial Complexes
- Walks on Simplicial Complexes and some known results
- Strongly log-concave polynomials
- Main results of the paper
- Some applications