Sampling Bases of a Matroid

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Outline

- Preliminaries
  - Linear Algebra
  - Simplicial Complexes
- Walks on Simplicial Complexes and some known results
- Recap
- Strongly log-concave polynomials and distributions
- Main result
- Secondary results

Recall

- Results
  - A randomised, polynomial time algorithm to sample a uniform random basis of a matroid $\implies$ FPRAS to count the number of bases of a matroid (Since Sampling $\leftrightarrow$ Counting).
  - Expansion of the bases exchange graph of a matroid is 1.

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  - A simplicial complex X on the ground set $[n]$ is a nonempty collection of subsets of $[n]$ that is downward closed. The elements of X are called **faces/simplices**. For any $1 \leq k \leq n$, we define the set of k-faces as,

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  - We defined a weighted simplicial complex (X, w) and a balanced weight function w.
  - For any $1 \leq k < d$, where d is the dimension of X, we saw the walks $P_k^{\wedge}$ and $P_{k+1}^{\vee}$. They are PSD and have the same non-zero eigenvalues.

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  - Link of a face $\tau \in X(k)$: $X_\tau = \{\sigma \setminus \tau \mid \sigma \in X, \sigma \supset \tau\}$.

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  - *Definition (Local Spectral Expanders)* [KO18][i]: For $\lambda \geq 0$, a pure d-dimensional weighted complex (X, w) is a λ-local-spectral-expander if for every $0 \leq k < d-1$, and for every $\tau \in X(k)$, we have $\lambda_2\left(\tilde{P}_{\tau,1}^{\wedge}\right) \leq \lambda$.

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Theorem 1 [KO18][ii]: Let (X, w) be a pure d-dimensional weighted 0-local spectral expander and let $0 \leq k < d$. Then, for all $-1 \leq i \leq k$, $P_k^{\wedge}$ has at most $|X(i)| \leq \binom{n}{i}$ eigenvalues of value $> 1 - \frac{i+1}{k+1}$, where for convenience, we set $X(-1) = \varphi$ and $\binom{n}{-1} = 0$. In particular, the second largest eigenvalue of $P_k^{\wedge}$ is at most $\frac{k}{k+1}$.

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(Strongly) log-concave polynomials

- A polynomial $p \in \mathbb{R}[x_1, x_2, \ldots, x_n]$ with non-negative coefficients is *log-concave* on a subset $K \subseteq \mathbb{R}^n_{\geq 0}$ if $\log p$ is a concave function at any point in K.

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- p is *strongly log-concave* on K if for any $k \geq 0$, and any sequence of integers $1 \leq i_1, \ldots, i_k \leq n$,

$$(\partial_{i_1} \ldots \partial_{i_k} p)(x_1, \ldots, x_n)$$

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  is log-concave on K.
- For proving the results, we'll only need strong log-concavity at $K = \{\mathbf{1}\}$.

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Lemma [AOV18]: If a d-homogeneous polynomial $p \in \mathbb{R}[x_1, \ldots, x_n]$ with non-negative coefficients is log-concave over $K \subseteq \mathbb{R}{>}\ 0^n$, then $\left(\nabla^2 p\right)(x)$ has at most one positive eigenvalue at all $x \in K$. The 0 polynomial is also considered to be log-concave.

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Proof: As p is log-concave on K, $\forall x \in K$,

$$\nabla^2 \log p = \frac{p \cdot \left(\nabla^2 p\right) - (\nabla p)(\nabla p)^\top}{p^2} \preccurlyeq 0.$$

As $p^2(x) \geq 0$ for any $x \in K$, $p \cdot \left(\nabla^2 p\right) - (\nabla p)(\nabla p)^\top \preccurlyeq 0$.

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Cauchy's Interlacing Theorem: For a symmetric matrix $A \in \mathbb{R}^{n \times n}$ and vector $v \in \mathbb{R}^n$, the eigenvalues of A interlace the eigenvalues of $A + vv^\top$. That is, for $B = A + vv^\top$,

$$\lambda_n(A) \leq \lambda_n(B) \leq \lambda_{n-1}(A) \leq \cdots \leq \lambda_2(B) \leq \lambda_1(A) \leq \lambda_1(B).$$

Hence, $p \cdot \left(\nabla^2 p\right)$ has at most one positive eigenvalue.

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Hence, $p \cdot \left(\nabla^2 p\right)$ has at most one positive eigenvalue.

As, p has non-negative coefficients and $K \subseteq R^n_{>0}$, $p(x) \geq 0 \; \forall x \in K$. Thus, $\nabla^2 p$ has at most one positive eigenvalue at any $x \in K$.

$\square$

Strongly log-concave distributions

- Let $\mu : 2^{[n]} \to \mathbb{R}^n_{\geq 0}$ be a probability distribution on the subsets of the set $[n]$. Then,

$$g_\mu(x) = \sum_{S \subseteq [n]} \mu(S) \cdot \prod_{i \in S} x_i.$$

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- If g_μ is strongly log-concave on $\{\mathbf{1}\}$, then we say μ is strongly log-concave at $\mathbf{1}$.

Markov Chain corresponding to μ

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Goal: To show that $\mathcal{M}_\mu$ mixes rapidly.

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Main Result

Theorem: Let $\mu : 2^{[n]} \to \mathbb{R}_{\geq 0}$ be a d-homogeneous strongly log-concave probability distribution (at $\mathbf{1}$). If P_μ denotes the transition probability matrix of $\mathcal{M}_\mu$ and $X(k)$ denotes the collection of size-k subsets of $[n]$ which are contained in some element of $\mathrm{supp}(\mu)$, then for every $0 \leq k \leq d-1$, P_μ has at most $|X(k)| \leq \binom{n}{k}$ eigenvalues of value $> 1 - \frac{k+1}{d}$. In particular, $\mathcal{M}_\mu$ has spectral gap at least $\frac{1}{d}$, and if τ is in the support of μ and $0 < \epsilon < 1$, the total variation mixing time of the Markov chain $\mathcal{M}_\mu$ started at τ is at most

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From now on, by strongly log-concave, we will mean strongly log-concave at $\{\mathbf{1}\}$.

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- Observe that $P_d^\vee$ is exactly the same as P_μ.

Strongly log-concave polynomials to Local Spectral Expanders

Let $p = \sum_S c_S x^S \in \mathbb{R}[x_1, \ldots, x_n]$ be a multiaffine, d-homogeneous, strongly log-concave polynomial with non-negative coefficients, where $x^S = \prod_{i \in S} x_i$.

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The pure d-dimensional, weighted simplicial complex (X^p, w) is defined as follows:

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- For all $S \in X^p(d)$, $w(S) = c_S$. For $d - 1 \geq k \geq 1$ and $\tau \in X(k)$, recursively define $w(\tau) = \sum_{\sigma \in X(k+1): \sigma \supset \tau} w(\sigma)$.

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We want to show that (X^p, w) is a 0-local spectral expander. Notice that, as p has non-negative coefficients, the weights of all faces in (X^p, w) are non-negative.

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Lemma 3: For any $0 \le k \le d$, and any simplex $\tau \in X^p(k), w(\tau) = (d-k)! \cdot p_\tau(\mathbf{1})$.

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Proof: By induction on $d - k$.

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Base case: If $\dim(\tau) = d$, then as p is multiaffine, $p_\tau = c_\tau$. Hence, true by definition of w.

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Base case: If $\dim(\tau) = d$, then as p is multiaffine, $p_\tau = c_\tau$. Hence, true by definition of w.

Induction step: Suppose the statement holds for all simplices $\sigma \in X^p(k+1)$ and fix a simplex $\tau \in X^p(k)$. Then by definition,

$$w(\tau) = \sum_{\substack{\sigma\in X^p(k+1):\\ \sigma\supset\tau}} w(\sigma) \overset{\text{(A)}}{=} (d-k-1)! \sum_{\substack{\sigma\in X^p(k+1):\\ \sigma\supset\tau}} p_\sigma(1) = (d-k-1)! \sum_{i\in X^p_\tau(1)} p_{\tau\cup i}(\mathbf{1})$$

$$\overset{\text{(B)}}{=} (d-k-1)! \sum_{i=1}^{n} \partial_i p_\tau(\mathbf{1}) \overset{\text{(C)}}{=} (d-k)! \cdot p_\tau(\mathbf{1}),$$

(A): Induction hypothesis. (B) As $\partial_i p_\tau = 0$, for $i \notin X^p_\tau(1)$,
(C): Euler's identity: For a d-homogeneous polynomial p, $d \cdot p(x) = \sum_{k=1}^n x_k \partial_k p(x)$. $\square$

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Proposition 1: Let $p \in \mathbb{R}[x_1, \ldots, x_n]$ be a multiaffine, d-homogeneous polynomial, strongly log-concave polynomial with non-negative coefficients. Then (X^p, w) is a 0-local-spectral-expander.

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Proof: Recall that, $\widetilde{P}^{\wedge}_{\tau,1}$ is the transition probability matrix of the non-lazy random walk on the link X^p_τ. We need to show that $\lambda_2\left(\widetilde{P}^{\wedge}_{\tau,1}\right) \leq 0$ for all $\tau \in X$.

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Proof: Recall that, $\widetilde{P}^{\wedge}_{\tau,1}$ is the transition probability matrix of the non-lazy random walk on the link X^p_τ. We need to show that $\lambda_2 \left(\widetilde{P}^{\wedge}_{\tau,1} \right) \leq 0$ for all $\tau \in X$. Fix a τ. Since

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$$\tilde{\nabla}^2 p_\tau = \frac{1}{d-k-1} \operatorname{diag}\left(\nabla p_\tau(\mathbf{1})\right)^{-1} \nabla^2 p_\tau(\mathbf{1}).$$

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$$\begin{aligned}
\implies \left(\tilde{\nabla}^2 p_\tau\right)(i,j) &= \frac{\left(\partial_i \partial_j p_\tau\right)(\mathbf{1})}{(d-k-1)\cdot\left(\partial_i p_\tau\right)(\mathbf{1})} \\
&= \frac{(d-k-2)!\left(\partial_i \partial_j p_\tau\right)(\mathbf{1})}{(d-k-1)!\cdot\left(\partial_i p_\tau\right)(\mathbf{1})} \\
&= \frac{w\left(\tau \cup \{i,j\}\right)}{w\left(\tau \cup \{i\}\right)} && \text{(from Lemma 3)} \\
&= \frac{w_\tau\left(\{i,j\}\right)}{w_\tau\left(\{i\}\right)} \quad = \widetilde{P}^{\wedge}_{\tau,1}(\{i\},\{j\})
\end{aligned}$$

$$\tilde{\nabla}^2 p_\tau \;=\; \frac{1}{d-k-1}\mathrm{diag}\left(\nabla p_\tau(\mathbf{1})\right)^{-1}\nabla^2 p_\tau(\mathbf{1}) \;=\; \widetilde{P}^{\wedge}_{\tau,1}.$$

Since p has non-negative coefficients, $\nabla p_\tau(\mathbf{1})$ has non-negative entries $\implies \mathrm{diag}\left(\nabla p_\tau(\mathbf{1})\right) \succcurlyeq 0$.

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Since $\nabla^2 p_\tau(\mathbf{1})$ has at most one positive eigenvalue, $\tilde{\nabla}^2 p_\tau$ has at most one positive eigenvalue. Therefore, $\widetilde{P}^{\wedge}_{\tau,1}$ has at most one positive eigenvalue and $\lambda_2\left(\widetilde{P}^{\wedge}_{\tau,1}\right) \leq 0$.

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Since g_μ is strongly log-concave, by Proposition 1, X^{g_μ} is 0-local-spectral-expander. Therefore by Theorem 1,

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Thus, for $\mathcal{M}_{\mu}$, $t_{\tau}(\epsilon) \leq d \cdot \log\left(\frac{1}{\epsilon \cdot \mu(\tau)}\right)$.

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Corollary of the main result: Sampling and counting bases of matroids

Let $M = ([n], \mathcal{I})$ be an arbitrary matroid on n elements of rank r.

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Note that to run $\mathcal{M}_\mu$ we only need an oracle to test whether a given set $S \subseteq [n]$ is an independent set of M. Therefore, with only polynomially many queries (in $n, r, \log \frac{1}{\epsilon}$) we can generate a random base of M.

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Corollary of the main result: Sampling and counting bases of matroids

Corollary: For any matroid $M = ([n], \mathcal{I})$ of rank r, any basis B of M and $0 < \epsilon < 1$, the mixing time of the Markov chain M_μ starting at B is at most

$$t_B(\epsilon) \leq r \log\left(n^r/\epsilon\right) \leq r^2 \log\left(n/\epsilon\right).$$

Proof: $t_B(\epsilon) \leq r \cdot \log\left(1/(\epsilon \cdot \mu(B))\right)$. A matroid of rank r on n elements has at most $\binom{n}{r} \leq n^r$ bases. Thus, $\mu(B) \leq n^r$. □

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By equivalence of approximate counting and approximate sampling for self-reducible problems [JVV86[v]] we have the following,

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Corollary: There is a randomized algorithm that for any matroid M on n elements with rank r given by an independent set oracle, and any $0 < \epsilon < 1$, counts the number of bases of M up to a multiplicative factor of $1 \pm \delta$ with probability at least $1 - \delta$ in time polynomial in $n, r, 1/\epsilon, \log(1/\delta)$.

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Outline

- Preliminaries
  - Linear Algebra
  - Simplicial Complexes
- Walks on Simplicial Complexes and some known results
- Recap
- Strongly log-concave polynomials and distributions
- Main Result
- Secondary Results

Bases Exchange Graph

The bases exchange graph of a matroid M, denoted by G_M is a graph that has a vertex for every basis of M and two bases B, B' are connected by an edge if $|B \Delta B'| = 2$.

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Mihail and Vazirani [MV86[vi]] conjectured that the bases exchange graph has expansion at least 1.

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Proving the conjecture

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As $\mathcal{M}_\mu$ is a reversible Markov chain, we can represent it as a weighted undirected graph H_M.

- The vertices of H_M correspond to bases of M.
- The weight of an edge between two bases τ, τ' is, $P_\mu(\tau, \tau') = P_\mu(\tau', \tau)$,

i.e. the (weighted) adjacency matrix of H_M is P_μ.

Bases Exchange Walk

Thus, by *Cheeger's inequality*,

$$\operatorname{cond}(H_M) \geq \frac{1-\lambda_2(P_\mu)}{2} \geq \frac{1-\left(1-\frac{1}{r}\right)}{2} = \frac{1}{2r}.$$

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For any non-empty set $S \subset \mathcal{B}$ such that $|S| \leq |\mathcal{B}|/2$. As P_μ is a stochastic matrix, $\operatorname{vol}(S) = |S|$.

$$\operatorname{cond}(H_M) \leq \operatorname{cond}(S) = \frac{\sum_{\tau\in S,\tau'\notin S} P_\mu(\tau,\tau')}{|S|} \overset{(A)}{\leq} \frac{\sum_{\tau\in S,\tau'\notin S}\frac{1}{2r}}{|S|} = \frac{\frac{1}{2r}|E(S,\overline{S})|}{|S|} = \frac{h(S)}{2r}.$$

(A): If $P_\mu(\tau,\tau') \neq 0$, then $|\tau\cap\tau'| = r-1$. $P_\mu(\tau,\tau') = \frac{1}{r}\cdot\frac{1}{|\{\sigma\in\mathcal{B} \,:\, \tau\cap\tau'\subset\sigma\}|} \leq \frac{1}{2r}$.

$\square$

A characterisation of strongly-log concave polynomials

Definition: A polynomial $p \in \mathbb{R}[x_1, \ldots, x_n]$ is said to be decomposable if there exists a nonempty subset $I \subsetneq [n]$ and nonzero polynomials $g \in \mathbb{R}[x_i : i \in I]$, $h \in \mathbb{R}[x_i : i \notin I]$ for which $f = g + h$. Otherwise, f is indecomposable.

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Theorem: Let $p \in \mathbb{R}[x_1, \ldots, x_n]$ be a d-homogeneous polynomial such that:

1. for any $0 \leqslant k \leqslant d - 2$ and any $(i_1, \ldots, i_k) \in [n]^k$, $\partial_{i_1} \cdots \partial_{i_k} p$ is indecomposable, and
2. for any $(i_1, \ldots, i_{d-2}) \in [n]^{d-2}$, the quadratic $\partial_{i_1} \ldots \partial_{i_{d-2}} p$ is either identically zero, or log-concave at $\mathbf{1}$.

Then p is strongly log-concave at $\mathbf{1}$.

A characterisation of strongly-log concave polynomials

Fact: [AOV18[vii]] A degree-d homogeneous polynomial $p \in \mathbb{R}[x_1, \ldots, x_n]$ with non-negative coefficients is log-concave over $\mathbb{R}^n_{>0}$ iff $(\nabla^2 p)(x)$ has at most one positive eigenvalue at all $x \in \mathbb{R}^n_{>0}$.

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Proof: By induction on the degree of p. If the degree of p is at most 2, then 2 implies that p is log-concave at 1.

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Proof: By induction on the degree of p. If the degree of p is at most 2, then 2 implies that p is log-concave at 1. Let q be a non-zero first or second order derivative of p. Then,

$$\nabla^2 \log q = \frac{q \cdot (\nabla^2 q) - (\nabla q)(\nabla q)^\top}{q^2} = \frac{-(\nabla q)(\nabla q)^\top}{q^2} \preccurlyeq 0.$$

Hence, all derivatives of p are also log-concave; so the theorem is true.

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Define the normalised Hessian $\tilde{\nabla}^2 p = \frac{1}{d-1} \mathrm{diag}(\nabla p(\mathbf{1}))^{-1} \nabla^2 p(\mathbf{1})$. Can be shown to be stochastic.

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We'll use the following inner product: $\langle \varphi, \psi \rangle_p = (d-1)\sum_{j=1}^{n} \varphi(j)\psi(j)(\partial_j p(\mathbf{1}))$. This gives $\|\varphi\|_p^2 = \langle \varphi, \varphi \rangle_p$. Can be shown that $(\tilde{\nabla}^2 p)$ is self-adjoint w.r.t. this inner product.

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$$\langle \varphi, (\tilde{\nabla}^2 p)\psi \rangle_p = \langle \varphi, \nabla^2 p(\mathbf{1})\psi \rangle \overset{(A)}{=} \frac{1}{d-2}\sum_{k=1}^n \langle \varphi, \nabla^2 p_k(\mathbf{1})\psi \rangle = \frac{1}{d-2}\sum_{k=1}^n \langle \varphi, \tilde{\nabla}^2 p_k \psi \rangle_{p_k} \quad (*)$$

(A): From Euler's identity.

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View $\nabla^2 p(\mathbf{1})$ as adjacency matrix of a weighted undirected graph with weights $\partial_i \partial_j p$ and $\tilde{\nabla}^2 p$ as the normalised adjacency matrix. As p is indecomposable, the corresponding graph $\tilde{\nabla}^2 p$ is connected, so $\lambda_2(\tilde{\nabla}^2 p) < 1 \implies \lambda_2(\tilde{\nabla}^2 p) \leq 0$.

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$$\mu \|\varphi\|_p^2 = \langle \varphi, \mu\varphi \rangle_p = \langle \varphi, (\tilde{\nabla}^2 p)\varphi \rangle_p \overset{\text{from } (*)}{=} \tfrac{1}{d-2} \textstyle\sum_{k=1}^n \langle \varphi, (\tilde{\nabla}^2 p_k)\varphi \rangle_{p_k}$$

Let $\varphi = \varphi_k^{\perp \mathbf{1}} + \varphi_k^{\mathbf{1}}$, where $\varphi_k^{\perp \mathbf{1}}$ is the component orthogonal to $\mathbf{1}$ and $\varphi_k^{\mathbf{1}} = \frac{\langle \varphi, \mathbf{1} \rangle_{p_k}}{\langle \mathbf{1}, \mathbf{1} \rangle_{p_k}} \mathbf{1}$ parallel to $\mathbf{1}$.

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From the induction hypothesis, p_k is strongly log-concave. Hence, $\nabla^2 p_k(1)$ and therefore, $\tilde{\nabla}^2 p_k$ have at most one positive eigenvalue. So, $\langle \varphi_k^{\perp \mathbf{1}}, (\tilde{\nabla}^2 p_k)\varphi_k^{\perp \mathbf{1}} \rangle_{p_k} \leqslant 0$.

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Thus,

$$\mu \|\varphi\|_p^2 \leqslant \frac{1}{d-2} \sum_{k=1}^n \langle \varphi_k^{\mathbf{1}}, (\tilde{\nabla}^2 p_k)\varphi_k^{\mathbf{1}} \rangle_{p_k} = \frac{1}{d-2} \sum_{k=1}^n \langle \varphi_k^{\mathbf{1}}, \varphi_k^{\mathbf{1}} \rangle_{p_k} = \frac{1}{d-2} \sum_{k=1}^n \frac{\langle \varphi, \mathbf{1} \rangle_{p_k}^2}{\langle \mathbf{1}, \mathbf{1} \rangle_{p_k}}.$$

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Now

$$\langle \mathbf{1}, \mathbf{1} \rangle_{p_k} = (d-2) \sum_{i=1}^{n} (\partial_i p_k(\mathbf{1})) = (d-2)(d-1) \cdot p_k(\mathbf{1})$$

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and

$$\begin{aligned}\langle \varphi, \mathbf{1}\rangle_{p_k} &= (d-2)\sum_{i=1}^n \varphi(i)(\partial_i p_k(\mathbf{1})) \\ &= (d-2)\sum_{i=1}^n \varphi(i)(\partial_i \partial_k p(\mathbf{1})) = (d-2)\sum_{i=1}^n \varphi(i)(\partial_k \partial_i p(\mathbf{1})) \\ &= (d-2)\cdot((\nabla^2 p(\mathbf{1}))\varphi)(k).\end{aligned}$$

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This gives,

$$\frac{\langle \varphi, \mathbf{1} \rangle_{p_k}}{\langle \mathbf{1}, \mathbf{1} \rangle_{p_k}} = \frac{1}{(d-1) \cdot p_k}((\nabla^2 p(\mathbf{1}))\varphi)(k) = ((\tilde{\nabla}^2 p)\varphi)(k) = \mu \cdot \varphi(k).$$

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Finally,

$$\begin{aligned} \mu \left\| \varphi \right\|_p^2 &\leqslant \frac{1}{d-2} \sum_{k=1}^{n} \frac{\langle \varphi, \mathbf{1} \rangle_{p_k}^2}{\langle \mathbf{1}, \mathbf{1} \rangle_{p_k}} = \frac{\mu}{d-2} \sum_{k=1}^{n} \varphi(k) \langle \varphi, \mathbf{1} \rangle_{p_k} \\ &= \mu \sum_{k=1}^{n} \varphi(k)((\nabla^2 p(\mathbf{1}))\varphi)(k) = \mu \langle \varphi, (\nabla^2 p(\mathbf{1}))\varphi \rangle = \mu \langle \varphi, (\tilde{\nabla}^2 p)\varphi \rangle_p = \mu^2 \left\| \varphi \right\|_p^2 . \end{aligned}$$

$\square$