Lifting with XOR

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Based on work done with Arkadev Chattopadhyay and Nikhil Mande

Dec 14, 2019

Statements made in these slides are for representational purposes and are not guaranteed to be entirely accurate.

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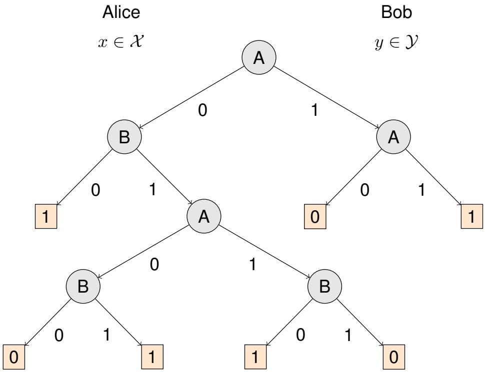
- In this talk, we focus on two parties (Alice and Bob) computing a total Boolean function.
- XOR functions feature in this talk because
  - They are structured enough to reason about.
  - There is enough mystery about them for them to be interesting.

A Communication Protocol



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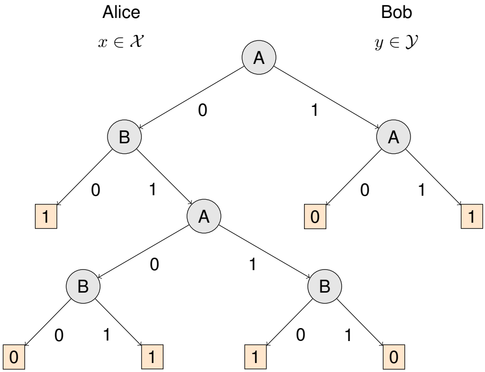

(x, y) is accepted

$\Leftrightarrow$

(x, y) reaches a 1-leaf.

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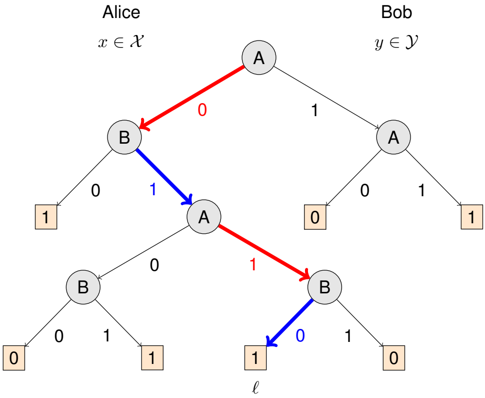

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Inputs that reach ℓ

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$\{x : x$ answers red$\}$

$\times$

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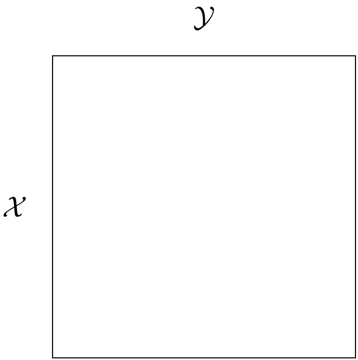
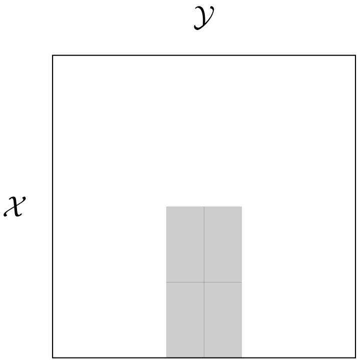
Rank



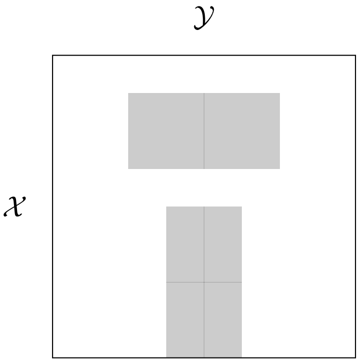
Building the truth table for the function computed by the protocol.

Rank



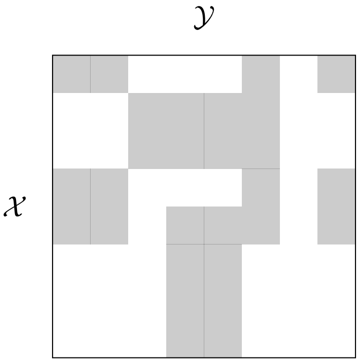
Inputs that reach leaf ℓ contribute a rank 1 matrix.

Rank



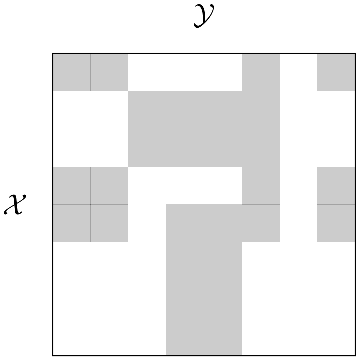
Inputs that reach leaves ℓ_1 or ℓ_2 form a rank ≤ 2 matrix.

Rank



Inputs that reach any 1 leaf form a rank $\leq 2^c$ matrix.

Rank



Cost c protocol for F

$$\Longrightarrow$$

M_F has rank $\leq 2^c$.

Protocol-Rank Equivalence?

Conjecture (Lovász Saks '88)

$$\exists \textit{ constant } \alpha \textit{ s.t. } D(F) \leq \log^{\alpha} \mathsf{rank}(F)$$

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Fun fact: LRC is True if you restrict the rank decomposition to be nonnegative.

A Randomized Communication Protocol

Alice

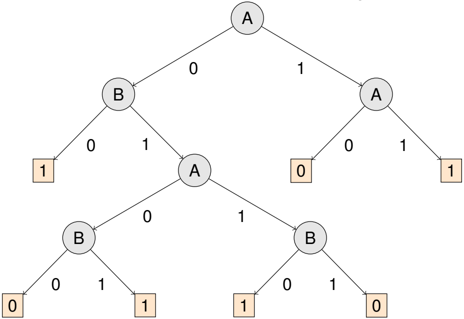
$x \in \mathcal{X}$

r_A, a string of random bits

Bob

$y \in \mathcal{Y}$

r_B, a string of random bits



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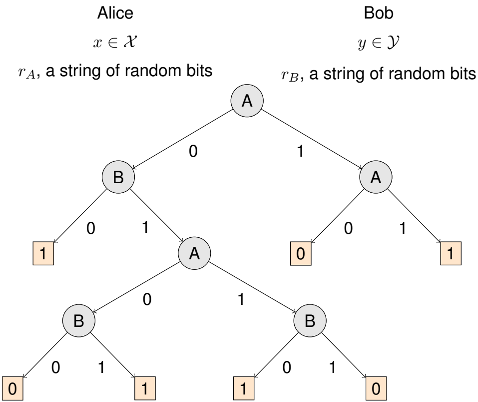

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$$=$$
$$\Pr[(x, y) \text{ reaches a } 1\text{-leaf}].$$

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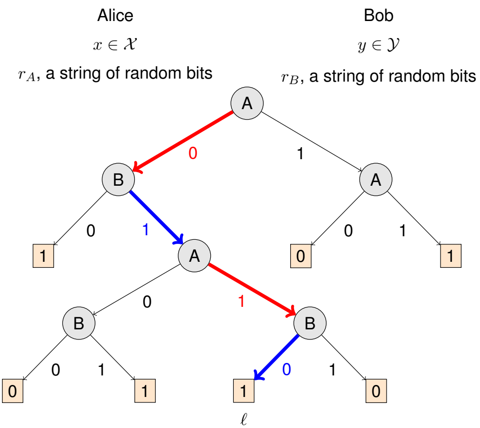

$\Pr[(x, y)$ is accepted$]$
$=$
$\Pr[(x, y)$ reaches a 1-leaf$]$.

$\Pr[(x, y)$ reaches $\ell]$
$=$
$\Pr_{r_A}[x$ answers red$]$
$\times$
$\Pr_{r_B}[y$ answers blue$]$.

Small Approximate Rank

$$\Pr_{r_B}[y \text{ answers blue}]$$

$$
\begin{array}{c c}
 & \begin{array}{cccc} 0 & .5 & 0 & .6 \end{array} \\
\Pr_{r_A}[x \text{ answers red}] \quad \begin{array}{c} .5 \\ .8 \\ 0 \\ 0 \end{array} & \left[\begin{array}{cccc} & .25 & & .3 \\ & .4 & & .48 \\ & & & \\ & & & \end{array}\right]
\end{array}
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$\Pr[(x, y) \text{ reaches } \ell]$ is a rank 1 matrix.

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$\mathrm{Pr}[(x, y)$ reaches $\ell]$ is a rank 1 matrix.

$\mathrm{Pr}[(x, y)$ is accepted$]$ is a rank $\leq 2^c$ matrix.

$$\boxed{\begin{matrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{matrix}}$$

M_F

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$$\log \mathsf{rank}_{1/3}(F) \leq c.$$

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Conjecture (ForgeGod '05, Lee Shraibman '07)

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For a randomized protocol, the number of bits exchanged in the worst case, R(f), is conjectured to be polynomially related to the following absurd formula:

min{rank(M'_f): $M'_f \in \mathbb{R}^{2^n \times 2^n}$, $(M_f - M'_f)_\infty \leq 1/3$}.

Figure: Screenshot from "Communication complexity - Wikipedia" (Dec '05)

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[Göös Jayram Pitassi Watson '17] showed that $\beta \geq 4$.

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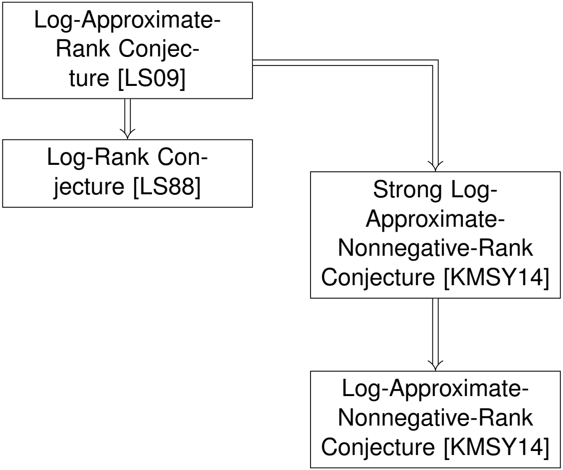
- [Kol Moran Shpilka Yehudayoff '14] did.

To Cache


Log-Approximate-Rank Conjecture [LS09]
Log-Rank Conjecture [LS88]
Strong Log-Approximate-Nonnegative-Rank Conjecture [KMSY14]
Log-Approximate-Nonnegative-Rank Conjecture [KMSY14]

To Cache

$$
\begin{tikzcd}[arrows=Rightarrow]
\begin{array}{c} R(F) \text{ vs.} \\ \log \mathsf{rank}_{1/3}(F) \end{array} \arrow[d] \arrow[dr] & \\
\begin{array}{c} D(F) \text{ vs.} \\ \log \mathsf{rank}(F) \end{array} & \begin{array}{c} R(F) \text{ vs.} \\ \log \mathsf{rank}^{+}_{1/3}(F) \end{array} \arrow[d] \\
 & \begin{array}{c} R(F) \text{ vs.} \\ \log \left(\mathsf{rank}^{+}_{1/3}(F) + \mathsf{rank}^{+}_{1/3}(\overline{F}) \right) \end{array}
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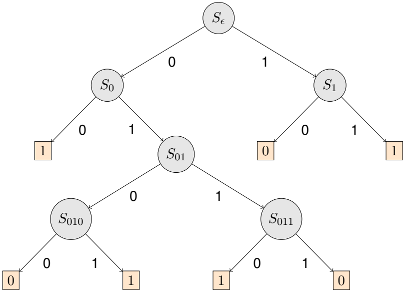
- But, parity can be computed easily.
- Expect to lift from Parity Decision Trees (allowed queries are parities, not just bits).

Parity Decision Trees (PDTs)

Hidden Input

$z \in \{0,1\}^n$



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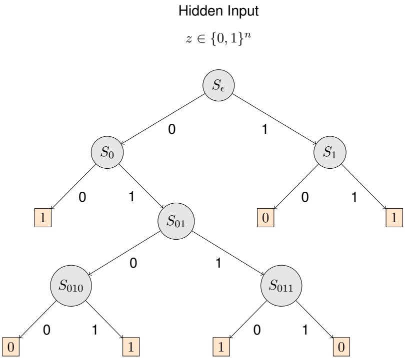

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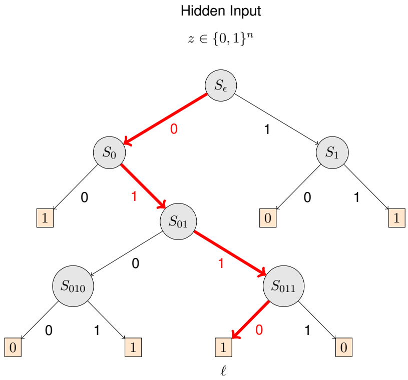

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OR: Hard for deterministic PDTs

- You have to reject just one input.
- Any leaf at depth d has 2^{-d} fraction of inputs.
- $\implies$ there must be a 0-leaf at depth n.

OR: Easy for randomized PDTs

- Randomly sample $S \subseteq_{\mathcal{U}} [n]$.
- Query $\oplus_S z$.

$$\Pr[\text{Query outputs } 0] = \begin{cases} 1 \text{ if } z = 0^n \\ 1/2 \text{ if } z \neq 0^n \end{cases}$$

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What other potentially fruitful properties do PDTs have?

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- Lifted Open Problem: Is there an XOR function easy for randomized communication but hard for P^{EQ} protocols?
  For general functions, this question was answered very recently. [Chattopadhyay Lovett Vinyals '19] exhibited a function with the separation.

Meet the PDT measures: Fourier Analysis

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$$AND(z_1, z_2, z_3) = \frac{1}{8} - \frac{1}{8}z_1 - \frac{1}{8}z_2 - \frac{1}{8}z_3 + \frac{1}{8}z_1z_2 + \frac{1}{8}z_1z_3 + \frac{1}{8}z_2z_3 - \frac{1}{8}z_1z_2z_3$$

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- Every leaf is an affine subspace in $\mathbb{F}_2$.
- For a function f computable by a depth-k PDT, $\mathsf{sp}(f) \leq 2^{2k}$ and $||\widehat{f}||_1 \leq 2^k$.
- For a function f computable by a depth-k RPDT, $||\widehat{f}||_{1,1/3} \leq 2^k$.

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Measure for f	Measure for $F = f \circ \text{XOR}$
$\mathsf{sp}(f)$	$\mathsf{rank}(F) = \mathsf{sp}(f)$
$\mathsf{sp}_{1/3}(f)$	$\mathsf{sp}_{1/3'}(f)/n \leq \mathsf{rank}_{1/3}(F) \leq \mathsf{sp}_{1/3}(f)$

If $RPDT(f) \leq \log^{O(1)} \mathsf{sp}_{1/3}(f)$, then the LARC is true for XOR functions!

Lifting with XOR

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Lifting with XOR

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Grolmusz [Grolmusz '97] conjectured: $R(F) \leq \log^{O(1)} ||\widehat{F}||_1$

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$PDT(f)$	$PDT(f)^{1/6} \leq D(F) \leq 2PDT(f)$

If the LRC is true for XOR functions, then
$PDT(f) \leq \log^{O(1)} \mathrm{sp}(f)$.

$\log \mathsf{sp}_{1/3'}(f)$, $\log \left|\left|\widehat{f}\right|\right|_{1,1/3}$: Potayto, Potahto+$\log n$

$$g(z) = \sum_S \hat{g}(S) z_S$$

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$$\mathbb{E}[g_{sample}(z)] = \sum_{S \subseteq [n]} \frac{|\hat{g}(S_1)|}{\left|\left|\widehat{g}\right|\right|_1} sgn(\hat{g}(S))z_S = \sum_S \hat{g}(S) z_S = g(z)$$

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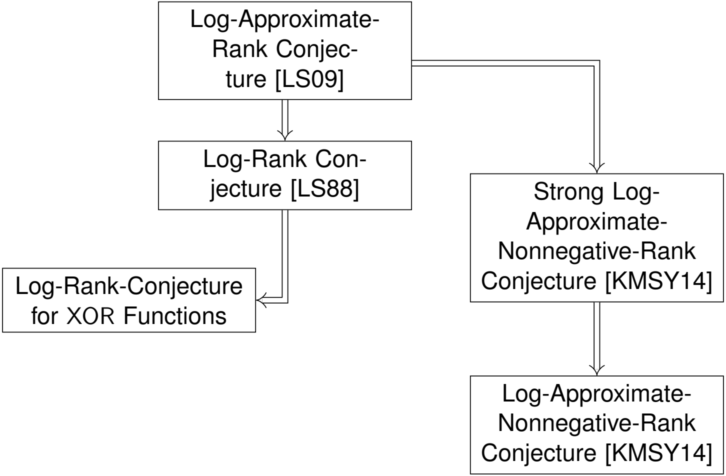
- LARC for XOR functions is "equivalent" to the corresponding ℓ_1-based conjecture for XOR functions. Implies Grolmusz' conjecture.

Take a Breather



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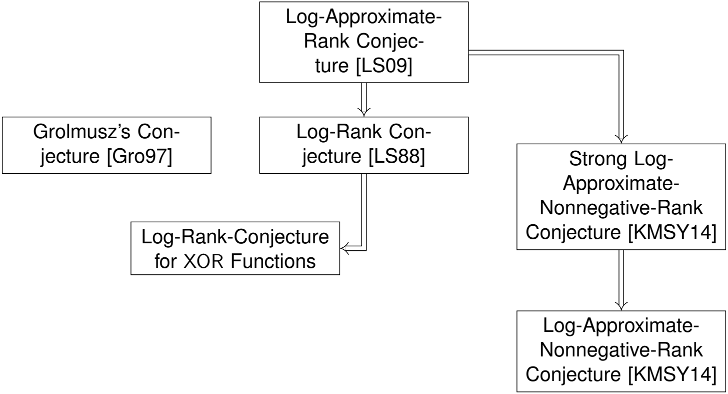

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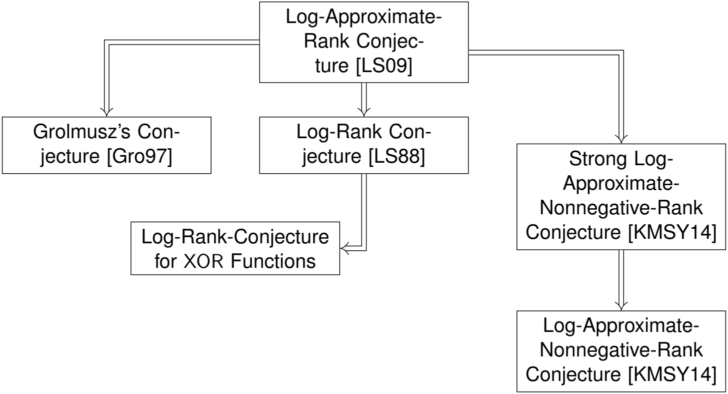

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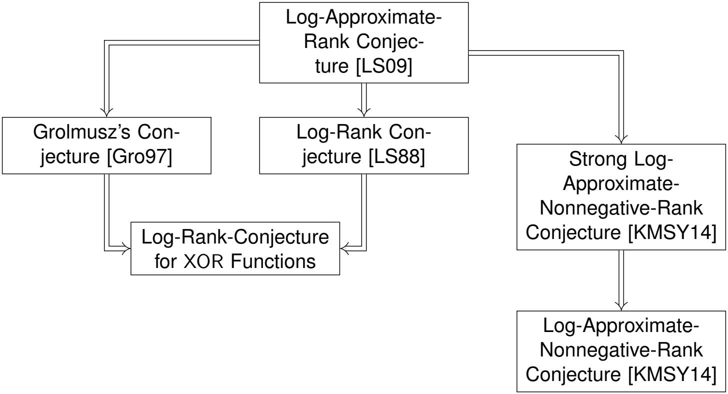

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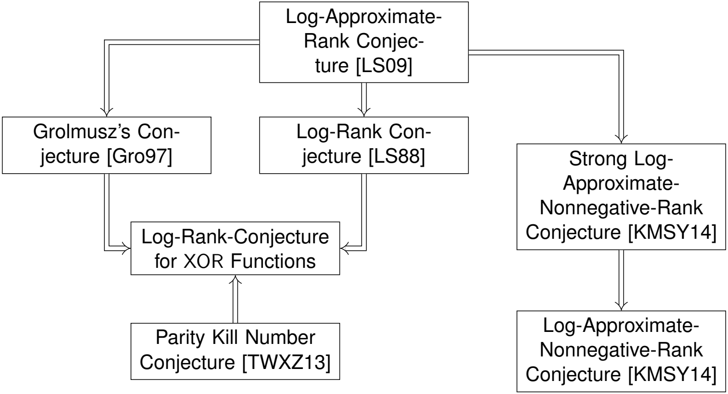

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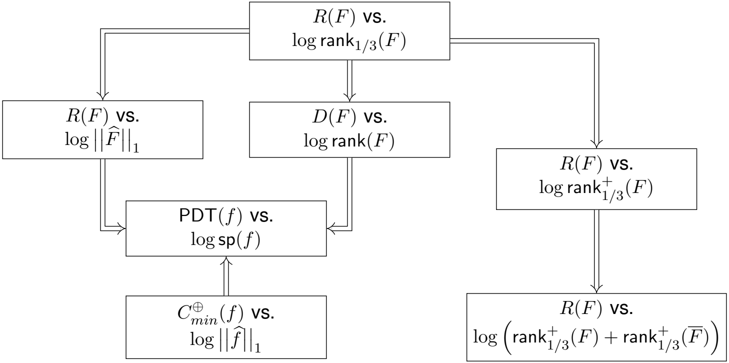

Is $RPDT(f) \leq \log \left|\left|\widehat{f}\right|\right|_1$?

Functions with small Fourier ℓ_1 norm:

- ANDs/affine subspaces.

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$$\begin{array}{cccc} & z_1 & z_2 & z_3 \\ S_1 & 0 & 0 & * \\ S_2 & 1 & * & 0 \\ S_3 & * & 1 & 1 \end{array}$$

Larger example

$$
\begin{array}{cccccc}
 & z_{1,2} & z_{1,3} & \cdots & z_{m-2,m} & z_{m-1,m} \\
S_1 & 0 & 0 & \cdots & * & * \\
S_2 & 1 & * & \cdots & * & * \\
\ldots & \ldots & \ldots & \ldots & \ldots & \\
S_{m-1} & * & * & \cdots & * & 0 \\
S_m & * & * & \cdots & 1 & 1
\end{array}
$$

Rephrased

$$\mathsf{SINK} : \{0,1\}^{\binom{m}{2}} \to \{0,1\}$$

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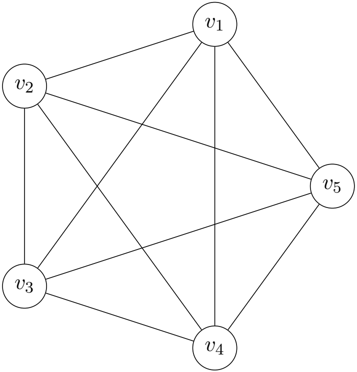

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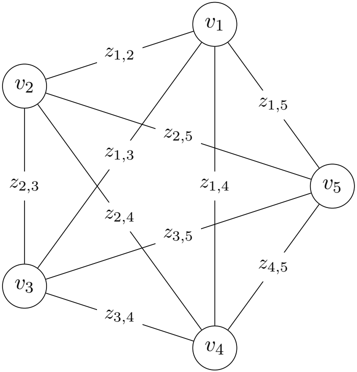

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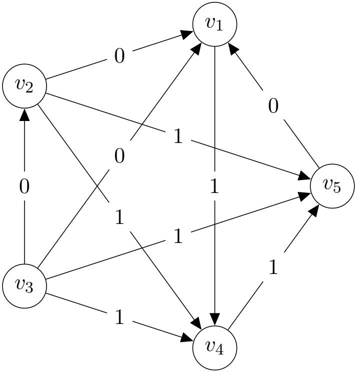

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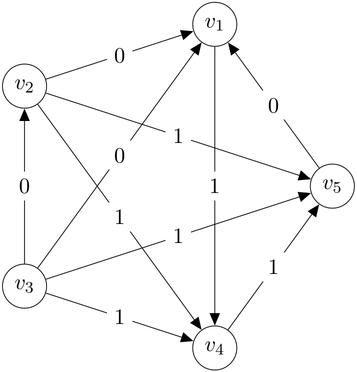
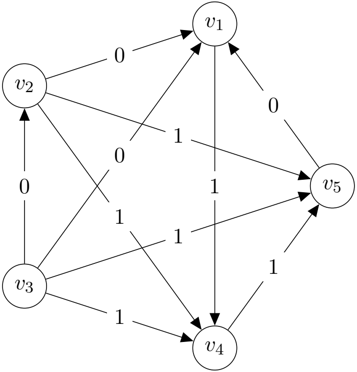


$\mathsf{SINK}(z) = 1$ iff there is a sink in the graph G_z.

Rephrased



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- $\left|\left|\widehat{\mathsf{SINK}}\right|\right|_1 \leq m$.
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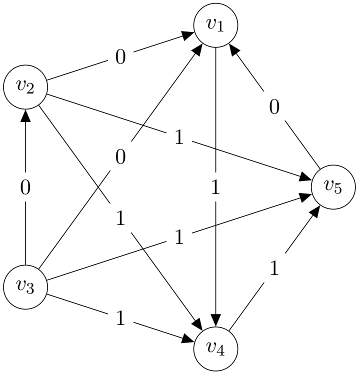
Rephrased

$$F := \mathsf{SINK} \circ \mathsf{XOR} : \{0,1\}^{\binom{m}{2}} \times \{0,1\}^{\binom{m}{2}} \to \{0,1\}$$



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$x \in \{0,1\}^{\binom{m}{2}}$

$z = x \oplus y$

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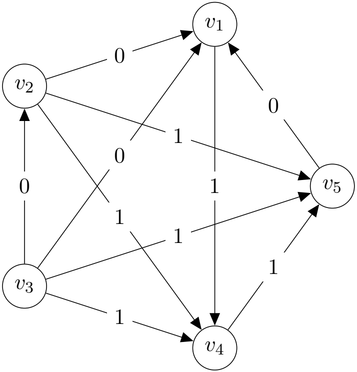

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Viewing it as a sum of equalities, $\mathsf{rank}^+_{1/3}(F) \leq m^{O(1)}$.

SINK is not easy

Theorem (Chattopadhyay Mande S '19)

$RPDT(\mathsf{SINK}) \geq \Omega(m)$, $R(\mathsf{SINK} \circ \mathsf{XOR}) \geq \Omega(m)$

SINK is not easy

Theorem (Chattopadhyay Mande S '19)

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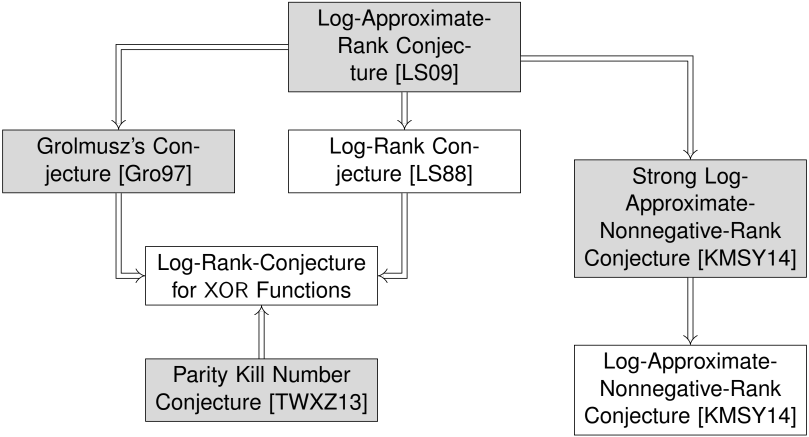
Sunken Conjectures



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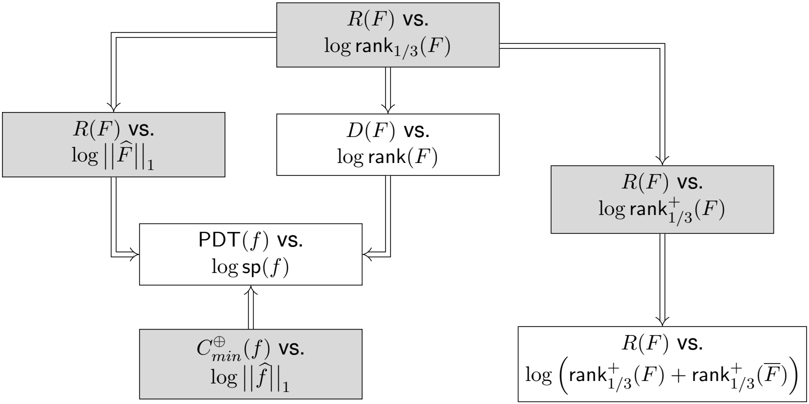

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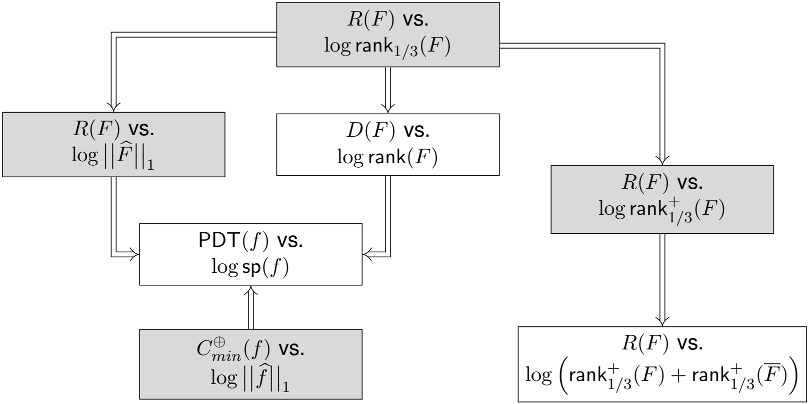

$\mathsf{rank}^+_{\epsilon}(\overline{\mathsf{SINK} \circ \mathsf{XOR}}) \geq 2^{\Omega(m)}$

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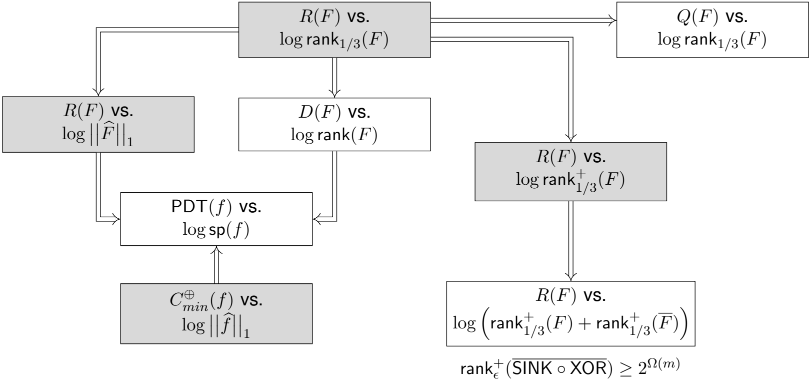
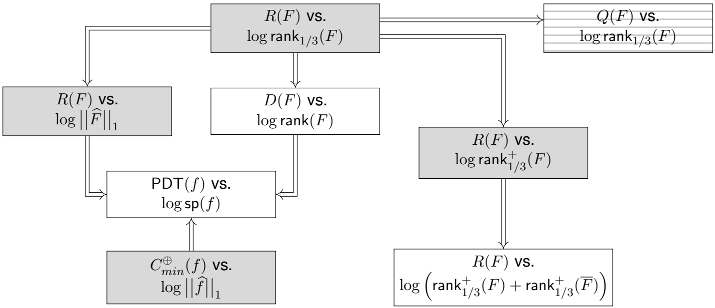


Sunken Conjectures

[Anshu Boddu Touchette '18, Sinha & de Wolf '18]



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About the Log-Approximate-Nonnegative-Rank Conjecture

- Can we have a function f wherein $f^{-1}(1)$ is a disjoint union of subcubes AND $f^{-1}(0)$ is a disjoint union of subcubes BUT f has large RPDT complexity?

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- The proof does not extend to disjoint unions of affine subspaces. Would be very interesting to settle this possibility.

Summary

- XOR functions behave well.
- PDTs are not well understood.
- Lots of juicy questions:
  - Are Randomized PDTs basically $\wedge$PDTs?
  - Can we close the avenue mentioned towards disproving the Log-Approximate-Nonnegative-Rank Conjecture?
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Thank you all for attending. I am open to questions and discussions.

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